NILPOTENTS IN FINITE ALGEBRAS*

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To the memory of Domingo Herrero, on the tenth aniversary of his passing

Abstract

We study the set of nilpotents t ($t^n = 0$) of a type II_1 von Neumann algebra $\mathcal{A}$ which verify that $t^{n-1} + t^*$ is invertible. These are shown to be all similar in $\mathcal{A}$. The set of all such operators, named by D.A. Herrero *very nice Jordan* nilpotents, forms a simply connected smooth submanifold of $\mathcal{A}$ in the norm topology. Nilpotents are related to systems of projectors, i.e. n-tuples $(p_1, ..., p_n)$ of mutually orthogonal projections of the algebra which sum 1, via the map

$$\varphi(t) = (P_{\ker t}, P_{\ker t^2} - P_{\ker t}, ..., P_{\ker t^{n-1}} - P_{\ker t^{n-2}}, 1 - P_{\ker t^{n-1}}).$$

The properties of this map, called the *canonical decomposition* of nilpotents in the literature, are examined.

Keywords: nilpotent operator, finite algebra.

1 Preliminaries

D.A. Herrero introduced the class of *very nice Jordan* operators as a solution to many aproximation problems in operator theory ([9]). For example, the problem of existence of similarity local cross sections. A very nice Jordan *nilpotent* operator t of order n of a C*-algebra $\mathcal{A}$ is an element $t \in \mathcal{A}$ such that $t^n = 0$, $t^{n-1} \neq 0$ and $t^{n-1} + t^*$ is invertible. The typical example of a very nice Jordan nilpotent occurs when $\mathcal{A} = M_n(\mathbb{C})$: consider the $n \times n$ Jordan cell Q_n, given by

$$Q_n = E_{1,2} + E_{2,3} + ... + E_{n-1,n}$$

where $E_{i,j}$ is the elementary matrix with 1 in the i, j entry and zero elsewhere. Let us transcribe the abstract characterization of these operators [9], Lemma 7.20:

Let $\mathcal{A}$ be a unital C-algebra and $t \in \mathcal{A}$ a nilpotent of order n. Then the following are equivalent:*

1. $t^{n-j} + t^{*j}$ *is invertible for all* $j = 1, ..., n-1$.
2. $t^{n-j} + t^{*j}$ *is invertible for some* $j = 1, ..., n-1$.
3. *There exists a faithful unital *-homomorphism* $\alpha : M_n(\mathbb{C}) \to \mathcal{A}$ *and an invertible element* $s \in \mathcal{A}$ *such that* $\alpha(Q_n) = sts^{-1}$.
4. *For every faithful representation* $\rho : \mathcal{A} \to B(H)$, *one has* $\ker \rho(t^j) = R(\rho(t^{n-j}))$ *for all* $j = 1, ..., n-1$.

*1991 Mathematics Subject Classification: 46L30, 46L05, 46L10.

In this paper we study the set of very nice Jordan nilpotents of a type II_1 algebra $\mathcal{A}$. We show that they are all similar. We also consider the subset consisting of very nice Jordan nilpotents which are partial isometries. These are shown to be all unitarily equivalent. Both sets are first considered in the norm topology. It is shown that they are smooth simply connected submanifolds of $\mathcal{A}$. Next they are considered in the strong operator topology. Here a stronger assumption on $\mathcal{A}$ is required, namely that $\mathcal{A}$ be a II_1 factor with the scaling trace property [12]. For these algebras we study fibrations relating very nice Jordan nilpotents to systems of pojections of $\mathcal{A}$ [7].

Let us establish some preliminary facts.

If $\mathcal{A}$ is a finite von Neumann algebra, one can find such nilpotents. This is clear for matrix algebras. If $\mathcal{A}$ is of type II_1, pick a projection p with $Tr(p) = 1/n$, where Tr is the center valued trace of $\mathcal{A}$. Then there exist equivalent projections $p = p_1, ..., p_n$ which are mutually orthogonal and sum 1. Let v_i, $i = 1, ..., n-1$ be partial isometries such that $v_i : p_{i+1} \sim p_i$, i.e. $v_i^* v_i = p_{i+1}$ and $v_i v_i^* = p_i$. Then

Lemma 1.1 *The element $a = \sum_{i=1}^{n} v_i$ is a very nice Jordan nilpotent and a partial isometry with kernel p and range $1 - p$.*

The proof follows from elementary computations. Also note that for each $0 \leq j \leq n-1$, $a^j p_n$ is a partial isometry with initial space p_n and final space p_{n-j}.

The so called canonical decomposition of a nilpotent operator will be useful. Given t a nilpotent of order n acting on H, one has the proper inclusions

$$\ker t \subset \ker t^2 \subset ... \subset \ker t^{n-1} \subset \ker t^n = H.$$

Put $H_1 = \ker t$, $H_2 = \ker t^2 \ominus \ker t, ..., H_j = \ker t^j \ominus \ker t^{j-1}$. The orthogonal subspaces $H_1, ..., H_n$ decompose H, and if one regards the $n \times n$ block matrix form of t in this decomposition, it is strictly upper triangular. We shall prefer to deal with the projections instead of the spaces, and will call the n-tuple $(p_{H_1}, ..., p_{H_n})$ the canonical decomposition of t, denoted by $\varphi(t)$. Note that if a is the operator defined above, then $\varphi(a) = (p_1, ..., p_n)$.

The canonical decomposition considered as a map was studied in [3], where the points of norm continuity of φ were characterized.

In this paper we shall consider the set $V_n(\mathcal{A})$ of very nice Jordan nilpotents of order n in the norm as well as in the strong operator topology. Continuity properties, both in the norm and strong operator setting, will follow from an explicit formula for $\varphi(t)$ for the case when t is very nice Jordan. In the strong operator topology though, one has to restrict to operators which are uniformly norm bounded. Let us end this section with the formula for φ.

Proposition 1.2 *Let $\mathcal{A}$ be a C^*-algebra and $t \in V_n(\mathcal{A})$. Then*

$$P_{\ker t^k} = t^{n-k} t^{*n-k} [(t^{n-k} + t^{*k})(t^{*n-k} + t^k)]^{-1},$$

for $k = 1, ..., n-1$.

Proof. Since $t^{n-k} + t^{*k}$ is invertible, and t^k has closed range, the operator given by the right hand expression has range $R(t^{n-k} t^{*n-k}) = R(t^{n-k}) = \ker(t^k)$. It remains to see that this expression defines a projection in $\mathcal{A}$. It is selfadjoint, because $t^{n-k}t^{*n-k}$ commutes with $(t^{n-k}+t^{*k})(t^{*n-k}+t^k)=t^{n-k}t^{*n-k}+t^{*k}t^k$. Moreover, $(t^{n-k}t^{*n-k})^2=t^{n-k}t^{*n-k}(t^{n-k}t^{*n-k}+t^{*k}t^k)=t^{n-k}t^{*n-k}(t^{n-k}+t^{*k})(t^{*n-k}+t^k)$. Therefore

$$\left(t^{n-k}t^{*n-k}[(t^{n-k}+t^{*k})(t^{*n-k}+t^k)]^{-1}\right)^2=(t^{n-k}t^{*n-k})^2[(t^{n-k}+t^{*k})(t^{*n-k}+t^k)]^{-2}=$$
$$t^{n-k}t^{*n-k}[(t^{n-k}+t^{*k})(t^{*n-k}+t^k)]^{-1}.$$

□

Note that $P_{\ker t^{k+1}\ominus\ker t^k}=P_{\ker t^{k+1}}-P_{\ker t^k}$. Therefore the statement above gives an explicit formula for $\varphi(t)$ in terms of the powers of t and t^*. In particular,

$$\varphi:V_n(\mathcal{A})\to P_n(\mathcal{A}):=\{(q_1,...,q_n)\in\mathcal{A}^n:q_iq_j=\delta_{i,j}q_i,q_i=q_i^*,q_1+...+q_n=1\}$$

is norm continuous.

2 Similarity and unitary equivalence in $V_n(\mathcal{A})$

In this section we shall prove that if $\mathcal{A}$ is a type II_1 von Neumann algebra, then all very nice Jordan nilpotents of $\mathcal{A}$ of order n are similar in $\mathcal{A}$. Also we will prove that all very nice Jordan nilpotents of order n, which additionally are partial isometries, are unitarily equivalent. The proof of the first fact proceeds in two steps.

Proposition 2.1 *Let $\mathcal{A}$ be a type II_1 von Neumann algebra with center valued trace τ. If $t\in V_n(\mathcal{A})$, then $\tau(\varphi(t)_i)=1/n$, $i=1,...n$.*

Proof. By the characterization of $V_n(\mathcal{A})$ transcribed before, there exists a unital injective *-homomorphism $\alpha:M_n(\mathbb{C})\to\mathcal{A}$ and an invertible element $s\in\mathcal{A}$ such that $\alpha(Q_n)=sts^{-1}$. The canonical decomposition of Q_n is $(E_{1,1},...,E_{n,n})$. Then $\varphi(sts^{-1})=\varphi(\alpha(Q_n))=(\alpha(E_{1,1}),...,\alpha(E_{n,n}))$, which are projections which are equivalent in $\mathcal{A}$. Therefore $\tau(\varphi(sts^{-1}))_i=1/n$, $i=1,...,n$. We claim that the n-tuples $\varphi(t)$ and $\varphi(sts^{-1})$ are unitarily equivalent in $\mathcal{A}$ (i.e. there exists a unitary element $u\in\mathcal{A}$ such that , $u\varphi(t)_iu^*=\varphi(sts^{-1})_i$, $i=1,...,n$) and this clearly ends our proof. Indeed, since the invertible group of $\mathcal{A}$ is connected, there is a norm continuous path of invertibles joining 1 and s. Since φ is norm continuous, this induces a norm continuous path joining $\varphi(t)$ and $\varphi(sts^{-1})$ in $P_n(\mathcal{A})$. In [7] it was shown that if the unitary group $U_{\mathcal{A}}$ is connected, then the connected components of $P_n(\mathcal{A})$ coincide with the unitary orbits of the elements of $P_n(\mathcal{A})$. It follows that $\varphi(t)$ and $\varphi(sts^{-1})$ are unitarily equivalent. □

Recall that we have fixed an n-tuple $(p_1,..,p_n)$ with $\tau(p_i)=1/n$ and an element $a\in V_n(\mathcal{A})$ with $\varphi(a)=(p_1,...,p_n)$. Let us define the following element of $\mathcal{A}$, for $t\in V_n(\mathcal{A})$,

$$\mu(t)=\sum_{i=0}^{n-1}t^ip_na^{*i}.$$

Proposition 2.2 *Suppose that $t\in V_n(\mathcal{A})$ with $\varphi(t)=(p_1,...,p_n)$. Then $\mu(t)$ is invertible and satisfies $t\mu(t)=\mu(t)a$.*

Proof. Suppose $\mathcal{A}$ acting in H. First note that $t^i p_n a^{*i}$ are operators with closed ranges which are in direct sum. Indeed, for $i = 0, ..., n-1$, $t^i p_n = (t^i + t^{*n-i})p_n$, because $R(p_n) = (\ker a^{n-1})^\perp = (\ker t^{n-1})^\perp = R(t)^\perp = \ker t^*$, i.e. $t^{*n-i}p_n = 0$ for $i < n$. Since $p_n a^{*i}$ is a partial isometry and $t^i + t^{*n-i}$ is invertible, this implies that $t^i p_n a^{*i}$ has closed range. Let us see that $R(t^i p_n a^{*i}) \cap R(t^j p_n a^{*j}) = \{0\}$ if $i \neq j$. Indeed, suppose $i > j$ and suppose that $t^i p_n \xi = t^j p_n \eta$, then $0 = t^{n-i+j} p_n \eta$, i.e. $p_n \eta \in \ker t^{n-(i-j)} \subset \ker t^{n-1}$. On the other hand $R(p_n) = (\ker t^{n-1})^\perp$. Therefore $p_n \eta = 0$. Therefore $\mu(t) = \sum_{i=0}^{n-1} t^i p_n a^{*i}$ has closed range. Moreover, it has trivial kernel: $\mu(t)\xi = 0$ implies, $t^i p_n a^{*i}\xi = (t^i + t^{*n-i})p_n a^{*i}\xi = 0$, which implies $p_n a^{*i}\xi = 0$, because $t^i + t^{*n-i}$ is invertible. Then ξ is orthogonal to the ranges of the partial isometries $a^i p_n$, which sum H, i.e. $\xi = 0$. Since the algebra $\mathcal{A}$ is finite, it follows that $\mu(t)$ is invertible.

Let us prove now that $t\mu(t) = \mu(t)a$. $t\mu(t) = \sum_{i=0}^{n-1} t^{i+1} p_n a^{*i} = \sum_{i=0}^{n-2} t^{i+1} p_n a^{*i}$. On the other hand, $\mu(t)a = \sum_{i=0}^{n-1} t^i p_n a^{*i} a$. We claim that $p_n a^{*i} a = p_n a^{*i-1}$ for $i \geq 1$ and $p_n a = 0$. These two facts clearly imply the equality $t\mu(t) = \mu(t)a$. The second fact is apparent, $R(a) = \ker a^{n-1} = \ker p_n$ ($R(p_n) = (\ker a^{*n-1})^\perp$). Let us prove that $p_n a^{*i} a = p_n a^{*i-1}$ for $i \geq 1$. Recall that $a = \sum_{j=1}^{n-1} v_j$ where v_j, are a partial isometries such that $v_i : p_{i+1} \sim p_i$. Then $a^i p_n = v_{n-i} v_{n-i+1} ... v_{n-1}$. It follows that

$$p_n a^{*i} a = v_{n-1}^* ... v_{n-i}^* (v_1 + ... + v_{n-1}) = v_{n-1}^* ... v_{n-i}^* v_{n-i},$$

because $v_{n-i}^* v_j = 0$ if $j \neq n-i$. The right hand term above equals

$$v_{n-1}^* ... v_{n-i}^* v_{n-i} = v_{n-1}^* ... v_{n-i+1} p_{n-i} = v_{n-1}^* ... v_{n-(i-1)}^* = p_n a^{*i-1},$$

and the proof is complete. □

With these two results we can prove our result on similarity.

Theorem 2.3 *Suppose that $\mathcal{A}$ is a type II_1 von Neumann algebra. Then all the elements of $V_n(\mathcal{A})$ are similar in $\mathcal{A}$.*

Proof. It suffices to show that if $t \in V_n(\mathcal{A})$, then t is similar to a. By the lemma above, $\varphi(t)$ and $\varphi(a)$ are unitarily equivalent. Indeed, $\varphi(t)_i$ and $\varphi(a)_i$ have the same trace, therefore there exists a partial isometries $w_i : \varphi(t)_i \sim \varphi(a)_i$, $i = 1, ..., n$. Since both the $\varphi(t)_i$ and $\varphi(a)_i$ sum 1, then $w = \sum_{i=1}^n w_i$ is a unitary operator such that $w\varphi(t)_i w^* = \varphi(a)$. Note that $w \ker t^i = \ker w t^i w^*$, therefore $w\varphi(t)w^* = \varphi(wtw^*)$. By the proposition above wtw^* and a are similar. Therefore t and a are similar. □

Remark 2.4 *Note that in particular, this result implies that the set $V_n(\mathcal{A})$ is connected when $\mathcal{A}$ is finite.*

Let us consider now elements of $V_n(\mathcal{A})$ which are *partial isometries*. Note that the element $a \in V_n(\mathcal{A})$ is one of these.

Theorem 2.5 *Let $\mathcal{A}$ be a type II_1 von Neumann algebra, $t \in V_n(\mathcal{A})$ and a as above. The following are equivalent:*

1. *t is a partial isometry (between $(\ker t)^\perp$ and $R(t)$).*

2. $t+t^{*n-1}$ *is a unitary element.*

3. *There exists* $u\in U_{\mathcal{A}}$ *such that* $t=uau^*$

Proof. Suppose first that $t+t^{*n-1}$ is unitary. Since $(\ker t)^{\perp}=R(t^*)=\ker t^{*n-1}$ it follows that $t=(t+t^{*n-1})P_{(\ker t)^{\perp}}$ is a partial isometry.

On the other hand, if t is a partial isometry, then t maps $\ker t^i\ominus\ker t^{i-1}$ onto $\ker t^{i-1}\ominus\ker t^{i-2}$. Indeed, $t^n=0$ implies that $t(\ker t^i)\subset\ker t^{i-1}$, and because t is very nice Jordan one has equality: pick $\xi\in\ker t^{i-1}=R(t^{n-i+1})$, $\xi=t^{n-i+1}\eta$, then if $\nu=t^{n-i}\eta$, $t\nu=\xi$ and $\nu\in\ker t^i$. Since t is isometric on $\ker t^i\ominus\ker t^{i-1}$ for $i\geq 2$, this implies that $t(\ker t^i\ominus\ker t^{i-1})=\ker t^{i-1}\ominus\ker t^{i-2}$ for $i\geq 2$. Therefore if $\varphi(t)=(q_1,...,q_n)$, t^lq_i is a partial isometry with initial space q_i and final space q_{i-l} for $l\leq i-1$ and 0 if $l\geq i$. In particular, since $q_n=1-P_{\ker t^{n-1}}=P_{R(t^{*n-1})}$, $t^{*n-1}=q_nt^{*n-1}$ is a partial isometry between q_1 and q_n. Therefore $t+t^{*n-1}$ is isometric between $P_{\ker t}\oplus q_1=1$ and $R(t)\oplus q_n=1$, i.e. a unitary.

Morever, if t is a partial isometry, then t is unitarily equivalent to a. In fact, t has an analogous description as a, $t=\sum_{i=2}^n tq_i$ with $tq_i:q_i\sim q_{i-1}$. By the same argument as in 2.3 there exists a unitary $u\in\mathcal{A}$ such that $u\varphi(t)u^*=\varphi(utu^*)=(uq_1u^*,...,uq_nu^*)=\varphi(a)=(p_1,...,p_n)$. Recall the map μ, and put $\mu(utu^*)$. As in 2.2, $\mu(utu^*)$ intertwines utu^* and a. Let us see that in this case $\mu(utu^*)$ is a unitary. Indeed, t^iq_n is, as shown above, a partial isometry between q_n and q_{n-i}. Then $ut^iq_nu^*=(utu^*)^ip_n$ is a partial isometry between $uq_nu^*=p_n$ and p_{n-i}. Then $(utu^*)^ip_na^{*i}$ is a partial isometry with final and initial space p_{n-i} for $i=1,...,n-1$. Therefore $\mu(utu^*)=\sum_{i=0}^{n-1}(utu^*)^ip_na^{*i}$ is a unitary operator.

□

Let us denote by $V_n^{\perp}(\mathcal{A})$ the set of very nice Jordan nilpotents of order n which are partial isometries.

These results imply that given $a\in V_n^{\perp}(\mathcal{A})$, one can regard the elements of $\mathcal{A}$ as n by n matrices with entries in $p\mathcal{A}p$, and replace a by a canonical model, which is unitarily equivalent to a, whose matrix looks like the matrix $Q_n\in M_n(\mathbb{C})$. We shall make use of this representation when examining the homotopy groups of these sets $V_n(\mathcal{A}),V_n^{\perp}(\mathcal{A})$ and $P_n(\mathcal{A})$ in section 4.

3 Norm continuous local cross sections

Regarding the way that the above results are proved, it follows that if one can exhibit an explicit (continuous, smooth) way to intertwine two systems of projections $(p_1,...,p_n)$ and $(q_1,...,q_n)$, then one would obtain a cross section from $V_n(\mathcal{A})$ to the invertible group $G_{\mathcal{A}}$ of $\mathcal{A}$, and for the unitary orbit of a , $V_n^{\perp}(\mathcal{A})$, as well . Let us briefly recall some facts from [7] concerning the geometry of systems of projections. Fix the n-tuple $\vec{p}=(p_1,...,p_n)\in P_n(\mathcal{A})$ and consider the map

$$\pi_{\vec{p}}:U_{\mathcal{A}}\to P_n(\mathcal{A}),\quad \pi_{\vec{p}}(u)=u\vec{p}u^*=(up_1u^*,...,up_nu^*).$$

This map has continuous local cross sections. Let $\mathcal{V}_{\vec{p}}=\{\vec{q}\in P_n(\mathcal{A}):s_{\vec{p}}(\vec{q}):=\sum_{i=1}^n q_ip_i\in G_{\mathcal{A}}\}$. Note that since $s_{\vec{p}}(\vec{p})=1$ and $G_{\mathcal{A}}$ is open in the norm topology it follows that $\mathcal{V}_{\vec{p}}$ is open in $P_n(\mathcal{A})$ (considered with the norm topology of $\mathcal{A}^n$). If $\vec{q} \in \mathcal{V}_{\vec{p}}$, then it is apparent that the invertible element $s_{\vec{p}}(\vec{q})$ intertwines $\vec{p}$ and $\vec{q}$, i.e. $s_{\vec{p}}(\vec{q})\vec{p} = \vec{q}s_{\vec{p}}(\vec{q})$. In order to obtain a *unitary* intertwiner one proceeds as follows: $s_{\vec{p}}(\vec{q})^* s_{\vec{p}}(\vec{q})$ conmmutes with p_i, $i = 1, ..., n$, therefore if we put $\sigma_{\vec{p}}(\vec{q})$ the unitary part of the polar decomposition $s_{\vec{p}}(\vec{q}) = \sigma_{\vec{p}}(\vec{q})(s_{\vec{p}}(\vec{q})^* s_{\vec{p}}(\vec{q}))^{1/2}$, one has

$$\sigma_{\vec{p}}(\vec{q})\vec{p}\sigma_{\vec{p}}(\vec{q})^* = \vec{q},$$

in other words,

$$\sigma_{\vec{p}} : \mathcal{V}_{\vec{p}} \to U_{\mathcal{A}}$$

is a continuous, in fact smooth, local cross section for $\pi_{\vec{p}}$. One can obtain local cross sections (neighbourhoods and maps) around any point in $P_n(\mathcal{A})$ by translating this one with unitaries. Combining this fact with the proof of the theorem of the previous section one obtains:

Proposition 3.1 *The map*

$$\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A}), \quad \pi_a(g) = gag^{-1}$$

is a principal bundle, with fibre equal to the group of invertible operators which commute with a.

Proof. Put $\vec{p} = \varphi(a)$. Consider the following map

$$\Lambda(t) = \sigma_{\vec{p}}(\varphi(t))\mu(\sigma_{\vec{p}}(\varphi(t))^* t \sigma_{\vec{p}}(\varphi(t)))$$

defined on the set $\{t \in V_n(\mathcal{A}) : \varphi(t) \in \mathcal{V}_{\vec{p}}\}$, which is open in $V_n(\mathcal{A})$. This map Λ is a continuous cross section for π_a on this subset of $V_n(\mathcal{A})$. Indeed, note that $\sigma_{\vec{p}}(\varphi(t))^* t \sigma_{\vec{p}}(\varphi(t))$ has the same canonical decomposition as a, because $\sigma_{\vec{p}}(\varphi(t))$ intertwines $\varphi(t)$ and $\varphi(a) = \vec{p}$. Therefore, by the proposition of the previous section, $\mu(\sigma_{\vec{p}}(\varphi(t))^* t \sigma_{\vec{p}}(\varphi(t)))$ intertwines a and $\sigma_{\vec{p}}(\varphi(t))^* t \sigma_{\vec{p}}(\varphi(t))$, and a simple computation shows that $\Lambda(t)$ intertwines a and t. Clearly it takes values in $G_{\mathcal{A}}$, and is continuous, therefore it defines a continuous local cross section for π_a. Another way of phrasing 2.3 above, is that the action of $G_{\mathcal{A}}$ is transitive on $V_n(\mathcal{A})$. Therefore this cross section can be carried over any point of $V_n(\mathcal{A})$ in a standard fashion. It is apparent that the fibre of π_a over a is the subgroup of invertibles which commute with a. Therefore $V_n(\mathcal{A})$ is in fact a homogeneous space, with norm continuous local cross sections, in particular it is a principal bundle. □

One can obtain the analogous result for $V_n^{\perp}(\mathcal{A})$ and the unitary group $U_{\mathcal{A}}$. The proof follows from the observation in the proof of 2.5, that $\mu(t)$ is in fact a unitary if $t \in V_n^{\perp}(\mathcal{A})$ with $\varphi(t) = \vec{p}$.

Proposition 3.2 *The map*

$$\pi_a : U_{\mathcal{A}} \to V_n^{\perp}(\mathcal{A}), \quad \pi_a(u) = uau^*$$

is a principal bundle, with fibre equal to the group of unitary operators which commute with a.

One can obtain more from the explicit fashion in which the map Λ is constructed. In [4] it was shown that if the cross section of a homogeneous space can be extended in a smooth manner to a neighbourhood of the algebra, then the homogeneous space becomes a complemented smooth submanifold of the algebra, and the map a smooth submersion. Let us cite this result, which is a more or less straightforward consequence of the inverse function theorem for Banach spaces:

Let $\mathcal{B}$ be a complex Banach algebra, $G_{\mathcal{B}}$ the Banach Lie group of invertible elements of $\mathcal{B}$, let $b \in \mathcal{B}$ and $\pi_b : G_{\mathcal{B}} \to S(b) = \{gbg^{-1} : g \in G_{\mathcal{B}}\}$ given by $\pi_b(g) = gbg^{-1}$. Then the following two conditions are equivalent:

1. *There exists a neighbourhood $b \in \mathcal{W} \subset \mathcal{B}$ and a smooth map $w_b : \mathcal{W} \to \mathcal{B}$ such that the restriction $w_b|_{S(b)}$ is a local cross section for π_b.*
2. *$\pi_b : G_{\mathcal{B}} \to S(b)$ is a smooth submersion and $S(b)$ is a complemented submanifold of $\mathcal{B}$.*

For a proof of this fact see [4].

Corollary 3.3 *The map $\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A})$ is a smooth submersion and $V_n(\mathcal{A})$ is a smooth complemented submanifold of $\mathcal{A}$.*

Proof. By the fact cited, it suffices to show that the cross section Λ has an extension to an open neighbourhood of a in $\mathcal{A}$. But this is apparent:

$$\Lambda(t) = \sigma_{\vec{p}}(\varphi(t))\mu(\sigma_{\vec{p}}(\varphi(t))^* t \sigma_{\vec{p}}(\varphi(t)))$$

is defined in terms of φ, which has an explicit expression given in section 1, and clearly extendible beyond $V_n(\mathcal{A})$, to the set of $x \in \mathcal{A}$ such that $x^j + x^{*n-j} \in G_{\mathcal{A}}$ for $j = 1, ..., n$, a set which is clearly open. This set must be eventually adjusted in order that the natural extension of $s_{\vec{p}}$ remains invertible.

□

Remark 3.4 *The restriction of the above map π_a to the unitary group has also unitary cross sections which can be smoothly extended to neighbourhoods of the norm topology of $\mathcal{A}$. It follows that $V_n^{\perp}(\mathcal{A})$ is a complemented submanifold of $V_n(\mathcal{A})$ (and of $\mathcal{A}$). This fact can also be obtained from the main result of [5], where it is shown that the unitary orbit of an element $b \in \mathcal{A}$ is a submanifold of $\mathcal{A}$ if b generates a finite dimensional C^*-algebra. Note that this is the case for a. In fact $C^*(a) \cong M_n(\mathbb{C})$.*

4 Homotopy type

In this section we examine the first homotopy groups of $V_n^{\perp}(\mathcal{A})$, $V_n(\mathcal{A})$ and $P_n^0(\mathcal{A})$. We make here again the assumption that $\mathcal{A}$ is of type II_1. In the previous sections we obtained that the maps

$$\pi_a : U_{\mathcal{A}} \to V_n^{\perp}(\mathcal{A}), \quad \pi_a(u) = uau^*$$

and

$$\pi_{\vec{p}} : U_{\mathcal{A}} \to P_n^0(\mathcal{A}), \quad \pi_{\vec{p}}(u) = (up_1u^*, \dots, up_nu^*)$$

are fibre bundles (homogeneous spaces) with fibre equal to (respectively) the unitary groups of $\{a\}' \cap \mathcal{A}$ and $\{p_1, ..., p_n\}' \cap \mathcal{A}$. In particular, $V_n^\perp(\mathcal{A})$ is homeomorphic to $U_\mathcal{A}/U_{\{a\}'\cap\mathcal{A}}$ and $P_n^0(\mathcal{A})$ is homeomorphic to $U_\mathcal{A}/U_{\{p_1,...,p_n\}'\cap\mathcal{A}}$. In order to carry on the computation of the homotopy groups, it suffices to consider the quotient maps instead of π_a and $\pi_{\vec{p}}$.

The element a provides a system of matrix units $e_{i,j} \in \mathcal{A}$, $i, j = 1, ..., n$, which satisfy $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$, $e_{i,j}^* = e_{j,i}$ and $e_{i,i} = p_i$. These elements enable one to identify $\mathcal{A}$ with $M_n(p\mathcal{A}p)$. Indeed, since $p \sim p_i$, the algebras $e_{i,i}\mathcal{A}e_{i,i}$ are isomorphic to $p\mathcal{A}p$ ($p = e_{1,1}$), via $x = e_{i,i}xe_{i,i} \mapsto e_{1,i}xe_{i,1}$. Then if $x \in \mathcal{A}$, $x \mapsto (x_{i,j})_{i,j}$, with $x_{i,j} = e_{1,i}xe_{j,1} \in p\mathcal{A}p$ yields the *-isomorphism. This isomorphism carries the subalgebra $\{a\}' \cap \mathcal{A}$ to the algebra $E_n(p\mathcal{A}p)$ of diagonal matrices which have the same element repeated along the diagonal. Indeed, if an element of $\mathcal{A}$ commutes with a, it commutes with the whole set of matrix units (which belong to the C*-algebra generated by a). The subalgebra $\{p_1, ..., p_n\}' \cap \mathcal{A}$ is carried to the algebra $D_n(p\mathcal{A}p)$ of diagonal matrices.

Therefore the study of the bundles π_a and $\pi_{\vec{p}}$ reduces to the study of the maps

$$\rho_1 : U_{M_n(p\mathcal{A}p)} \to U_{M_n(p\mathcal{A}p)}/U_{E_n(p\mathcal{A}p)}$$

and

$$\rho_2 : U_{M_n(p\mathcal{A}p)} \to U_{M_n(p\mathcal{A}p)}/U_{D_n(p\mathcal{A}p)}.$$

We shall need the following result, which is based on results from [6],[8] and [13], where it is shown that if M is a type II_1 von Neumann algebra, then $\pi_1(U_M, 1) \cong \mathcal{Z}(M)_{sa}$ the set of selfadjoint elements of the center of M, regarded as an additive group. Note that if $N \subset M$ is an inclusion of von Neumann algebras, then $\mathcal{Z}(M) \subset \mathcal{Z}(N)$.

Lemma 4.1 *Let $N \subset M$ be an inclusion of von Neumann algebras of type II_1, and denote by $\imath$ the inclusion map $\imath : U_N \hookrightarrow U_M$. Then the group homomorphism $\imath^* : \pi_1(U_N) \to \pi_1(U_M)$ identifies with the map $\tau_M|_{\mathcal{Z}(N)_{sa}} : \mathcal{Z}(N)_{sa} \to \mathcal{Z}(M)_{sa}$, where τ_M is the center valued trace of M.*

Proof. The group isomorphism between $\pi_1(U_M, 1)$ and the additive group $\mathcal{Z}(M)_{sa}$ is implemented as follows ([6], [8]):

$\pi_1(U_M, 1)$ is generated by classes of loops of the form $t \mapsto e^{it\pi q}$, where q ranges over all projections of M. The class of this loop is mapped to the element $\tau(q)$, where τ is the center valued trace of M.

The group $\pi_1(U_N, 1)$ is generated by the classes of the loops $t \mapsto e^{itr}$ for r a projection in N, which identifies with the element $\tau_N(r)$. The image of the class of this same loop under $\imath^*$ identifies with $\tau_M(r)$. Since $\mathcal{Z}(M) \subset \mathcal{Z}(M)$, the assignment $\tau_N(r) \mapsto \tau_M(r)$ is just the restriction of τ_M to $\mathcal{Z}(N)_{sa}$.

□

As an inmediate corollary, we obtain

Corollary 4.2 *With the above notations,($\mathcal{A}$ of type II_1), $\pi_k(V_n^\perp(\mathcal{A}), a)$ is trivial for $k = 0, 1, 2$.*

Proof. Clearly $M_n(p\mathcal{A}p)$ and $E_n(p\mathcal{A}p)$ are von Neumann algebras of type II_1 with the same center. Therefore in the the lemma above, $\imath^*$ is the identity. In the homotopy exact sequence of the fibration $\rho_1 : U_{M_n(p\mathcal{A}p)} \to U_{M_n(p\mathcal{A}p)}/U_{E_n(p\mathcal{A}p)} \cong V_n^\perp(\mathcal{A})$, one has

$$\ldots \to \pi_2(V_n^\perp(\mathcal{A}), a) \to \pi_1(U_{E_n(p\mathcal{A}p)}, 1) \xrightarrow{\imath^*} \pi_1(U_{M_n(p\mathcal{A}p)}, 1) \to$$

$$\to \pi_1(V_n^\perp(\mathcal{A}), a) \to \pi_0(U_{M_n(p\mathcal{A}p)}) = 0.$$

Since $\imath^*$ is an isomorphism, it follows that $\pi_2(V_n^\perp(\mathcal{A}), a) = \pi_1(V_n^\perp(\mathcal{A}), a) = 0$. We had already seen that $V_n^\perp(\mathcal{A})$ is connected. □

Consider now the inclusion $D_n(p\mathcal{A}p) \subset M_n(p\mathcal{A}p) = \mathcal{A}$.

Corollary 4.3 *With the above notations,($\mathcal{A}$ of type II_1), $P_n^0(\mathcal{A})$ is simply connected and*

$$\pi_2(P_n^0(\mathcal{A}), \vec{p}) \simeq (p\mathcal{Z}(\mathcal{A})_{sa})^{n-1}.$$

Proof. In this case, the inclusion to consider is $D_n(p\mathcal{A}p) \subset M_n(p\mathcal{A}p)$. The center of $D_n(p\mathcal{A}p)$ consists of diagonal matrices with entries in $\mathcal{Z}(p\mathcal{A}p)$. By the lemma above, the inclusion $U_{D_n(p\mathcal{A}p)} \hookrightarrow U_\mathcal{A}$ at the π_1-level is given by the map

$$(a_1, ..., a_n) \mapsto 1/n(a_1 + ... + a_n),$$

where the n-tuple $(a_1, ..., a_n)$ is identified with the diagonal matrix with such entries (in $\mathcal{Z}(p\mathcal{A}p)_{sa}$). In [13] Schröeder proved the if M is of type II_1, then $\pi_2(U_M, 1) = 0$. The exact sequence of the bundle ρ_2 is

$$\ldots 0 = \pi_2(U_\mathcal{A}, 1) \to \pi_2(P_n^0(\mathcal{A}), \vec{p}) \to \pi_1(U_{D_n(p\mathcal{A}p)}, 1) \to \pi_1(U_\mathcal{A}, 1) \to \pi_1(P_n^0, \vec{p}) \to 0.$$

The homomorphism $\pi_1(U_{D_n(p\mathcal{A}p)}, 1) = (\mathcal{Z}(p\mathcal{A}p)_{sa})^n \to \pi_1(U_\mathcal{A}, 1)$, $(a_1, ..., a_n) \mapsto 1/n(a_1 + ... + a_n)$ is clearly onto, which implies that $\pi_1(P_n^0(\mathcal{A})) = 0$. Its kernel equals $\pi_2(P_n^0(\mathcal{A}))$. This kernel clearly identifies with $(p\mathcal{Z}(\mathcal{A})_{sa})^{n-1}$. □

The result above, which states that the connected component $P_n^0(\mathcal{A})$ of $\vec{p} \in P_n(\mathcal{A})$ (and of every system of n projections consisting of *equivalent* projections) has trivial π_0 and π_1, holds in a weaker form for the unitary orbits (connected components) of arbitrary system of projections. In order to prove it, we need the following lemma, which was proved in [2] (lemma 6.2):

Lemma 4.4 *Let M be a type II_1 von Neumann algebra with center valued trace τ, and $p \in M$ a projection. Consider the map $\jmath : U_{pMp} \to U_M$ given by $\jmath(w) = w + 1 - p$. Then the image of*

$$\jmath^* : \pi_1(U_{pMp}, p) \to \pi_1(U_M, 1) \cong \mathcal{Z}(M)_{sa}$$

consists of the selfadjoint multiples of $\tau(p)$, i.e. $\{\tau(p)c : c \in \mathcal{Z}(M)_{sa}\}$.

Proposition 4.5 *Let $\vec{q} = (q_1, ..., q_n) \in P_n(\mathcal{A})$ be a system of projections of the II_1 von Neumann algebra $\mathcal{A}$. Then the connected component of $\vec{q}$ in $P_n(\mathcal{A})$ has trivial π_1 group.*

Proof As noted above, the connected component of $\vec{q} \in P_n(\mathcal{A})$ coincides with the unitary orbit of $\vec{q}$ [7]. Consider the principal bundle

$$\pi_{\vec{q}} : U_{\mathcal{A}} \to \{u\vec{q}u^* : u \in U_{\mathcal{A}}\}, \quad \pi_{\vec{q}}(u) = u\vec{q}u^*.$$

The fibre of this bundle is the unitary group of the relative commutant $\{q_1, ..., q_n\}' \cap \mathcal{A}$, i.e. $U_{q_1\mathcal{A}q_1 \oplus \ldots \oplus q_n\mathcal{A}q_n}$, which identifies with the product $U_{q_1\mathcal{A}q_1} \times \ldots \times U_{q_n\mathcal{A}q_n}$. Therefore the unitary orbit of $\vec{q}$ is homeomorphic to the quotient $U_{\mathcal{A}}/U_{q_1\mathcal{A}q_1} \times \ldots \times U_{q_n\mathcal{A}q_n}$. By the lemma above, the image of

$$\imath^* : \pi_1(U_{q_1\mathcal{A}q_1} \times \ldots \times U_{q_n\mathcal{A}q_n}, 1) \to \pi_1(U_{\mathcal{A}}, 1)$$

induced by the inclusion $\imath : U_{q_1\mathcal{A}q_1} \times \ldots \times U_{q_n\mathcal{A}q_n} \hookrightarrow U_{\mathcal{A}}$ contains the selfadjoint multiples of $\tau(q_i)$, $i = 1, ..., n$. Since these add up to 1, the image of $\imath^*$ is $\mathcal{Z}(\mathcal{A})_{sa}$, i.e. $\imath^*$ is onto. Then $\pi_1(U_{\mathcal{A}}/U_{q_1\mathcal{A}q_1 \oplus \ldots \oplus q_1\mathcal{A}q_1}, [1])$ is trivial (using the exact sequence of the fibre bundle $\pi_{\vec{q}}$).

$\square$

Remark 4.6 *The above result implies that any projection $r \in \mathcal{A}$ is unitarily equivalent to a projection which is diagonal with respect to the decomposition $(q_1, ..., q_n)$. Indeed, given any projection r, $\tau(r)$ equals the trace of some projection r' in $q_1\mathcal{A}q_1 \oplus \ldots \oplus q_n\mathcal{A}q_n$. Then r is unitarily equivalent to r', which is a diagonal projection.*

Let us turn now our attention to the similarity orbit of a, i.e. the set $V_n(\mathcal{A})$. The fibre bundle $\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A})$, $\pi_a(g) = gag^{-1}$ identifies with the quotient map of the invertible group $G_{\mathcal{A}}$ of $\mathcal{A}$ and the invertible group of the Banach algebra $\mathcal{T}$ of all elements of $\mathcal{A}$ which commute with a. A straightforward computation shows that (under the identification $\mathcal{A} \cong M_n(p\mathcal{A}p)$) $\mathcal{T}$ consists of matrices which have zeros below the diagonal, and are constant on the main diagonal and on the diagonals above: $b = (b_{i,j}) \in \mathcal{T}$ if $b_{i,j} = 0$ for $i > j$ and $b_{i,j} = b_{i+l,j+l}$, for $i \leq j$. Clearly the invertible group $G_{\mathcal{T}}$ consists of elements with $b_{1,1}$ invertible in $p\mathcal{A}p$.

Proposition 4.7 *Let $\mathcal{A}$ be a type II_1 von Neumann algebra. Then $V_n(\mathcal{A})$ is simply connected*

Proof. In an arbitrary von Neumann algebra, the invertible group is homotopically equivalent to the unitary group, via the polar decomposition (the set of positive invertible elements is convex). If $\alpha(t)$, $t \in [0, 1]$, is a curve in $G_{\mathcal{A}}$ whose endpoints are unitary elements, it can be continuously deformed to a curve $\alpha'(t)$ of unitaries (keeping the endpoints fixed). Let $\gamma(t)$ be loop in $V_n(\mathcal{A})$, with $\gamma(0) = \gamma(1) = a$. Since $\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A})$ is a fibre bundle, there exists a curve $\alpha(t) \in G_{\mathcal{A}}$ with $\alpha(0) = 1$ such that $\alpha(t)a\alpha(t)^{-1} = \gamma(t)$. Note that $\alpha(1)$ lies in $\mathcal{T}$. By the remark above, it is clear that $G_{E_n(p\mathcal{A}p)}$ is a strong deformation retract of $G_{\mathcal{T}}$. For example, consider the deformation $F_t((b_{i,j}))$ which multiplies by $t \in [0, 1]$ the entries above the diagonal, and leaves de diagonal entries fixed. Furthermore, since $U_{E_n(p\mathcal{A}p)}$ is a strong deformation retract of $G_{E_n(p\mathcal{A}p)}$, the curve $\alpha(t)$ can be continuously deformed to another curve, say again $\alpha(t)$, with $\alpha(0) = 1$ and $\alpha(1)$ a unitary element of $E_n(p\mathcal{A}p)$, the commutant of a. It follows that the original curve $\gamma(t)$ can be deformed to the loop $\alpha'(t)a\alpha'(t)^* \in V_n^{\perp}(\mathcal{A})$. Now 4.2 above implies that this loop can be deformed to the constant loop. Therefore $\pi_1(V_n(\mathcal{A}), a) = 0$.

□

Now we consider the fibration properties of φ

Proposition 4.8 *The map $\varphi : V_n(\mathcal{A}) \to P_n^0(\mathcal{A})$ is a fibration if $\mathcal{A}$ is of type II_1.*

Proof. Consider the following diagramm

$$\begin{array}{ccc} G_{\mathcal{A}} & \xrightarrow{\pi_a} & V_n(\mathcal{A}) \\ & \searrow \pi & \downarrow \varphi \\ & & P_n^0(\mathcal{A}). \end{array}$$

The diagonal arrow π is given by $\pi(g) = GS(gp_1g^{-1}, ..., gp_ng^{-1})$, where GS is a process of orthonormalization of the (non orthogonal) n-tuple $(gp_1g^{-1}, ..., gp_ng^{-1})$, called in [3] the Gram-Schmidt map. It is defined as follows: denote by $Q_n(\mathcal{A})$ the set of n-tuples $\vec{r} = (r_1, ..., r_n)$ of idempotents of $\mathcal{A}$ such that $r_ir_j = 0$ if $i \neq j$ and $r_1 + ... + r_n = 1$, put

$$GS_1(\vec{r}) = P_{R(r_1)}$$

for $k \geq 2$

$$GS_k(\vec{r}) = P_{R(r_1+...+r_k)} - P_{R(r_1+...+r_{k-1})}$$

and

$$GS(\vec{r}) = (GS_1(\vec{r}), ..., GS_n(\vec{r})).$$

This map is continuous (and smooth). It has an explicit form if one uses the well known formula, for r an idenpotent of a C*-algebra:

$$P_{R(r)} = rr^*(1 - (r - r^*)^2)^{-1}.$$

It is straightforward to verify that the diagram commutes [3]. Let us prove that $\pi : G_{\mathcal{A}} \to P_M^0(\mathcal{A})$ is a fibre bundle. First note that the fibre $\pi^{-1}(\vec{p})$ over $\vec{p}$ is a group. Indeed, it consists of the elements $g \in G_{\mathcal{A}}$ such that $GS(gp_1g^{-1}, ..., gp_ng^{-1}) = (p_1, ..., p_n)$, i.e.

$$R(g(p_1 + ... + p_k)g^{-1}) = g(R(p_1 + ... + p_k)) = R(p_1 + ... + p_k)$$

for $k = 1, ..., n-1$, a rule which clearly defines a subgroup of $G_{\mathcal{A}}$. On the other hand, π has local cross sections: if $\vec{q}$ is close to $\vec{p}$ in $P_n^0(\mathcal{A})$, then $s_{\vec{p}}(\vec{q}) = q_1p_1 + ... + q_np_n$ is an invertible element which intertwines $\vec{q}$ and $\vec{p}$, and $GS(s_{\vec{p}}(\vec{q})\vec{p}s_{\vec{p}}(\vec{q})^{-1}) = GS(\vec{q}) = \vec{q}$.

Therefore in the diagram above both the horizontal and diagonal arrows are fibre bundles. It follows by an elementary argument that the vertical arrow φ has the homotopy lifting property, and therefore is a fibration.

□

Remark 4.9 *Let us denote by $Q_n^0(\mathcal{A})$ the set of system of idempotents whose ranges are equivalent in $\mathcal{A}$. In [3] it was proven that $GS : Q_n^0(\mathcal{A}) \to P_n^0(\mathcal{A})$ is a homotopy equivalence. It follows that the results obtained for the homotopy groups of $P_n^0(\mathcal{A})$ hold for $Q_n^0(\mathcal{A})$.*

5 The canonical decomposition in the strong topology

In this section we shall regard the sets $V_n(\mathcal{A})$ and $V_n^\perp(\mathcal{A})$ with the strong operator topology of $\mathcal{A}$. There is the problem though, that $V_n(\mathcal{A})$ is not a bounded set, a fact which will trouble the strong continuity of φ, which is crucial in our exposition. For a constant $C > 0$, let $V_n^C(\mathcal{A})$ denote the set of $t \in V_n(\mathcal{A})$ such that $\|t\| \leq C$. The first result is certainly well known, we include a proof because we could not find a reference for it.

Lemma 5.1 *Let $\mathcal{A}$ be a finite von Neumann algebra, then the inversion map $g \mapsto g^{-1}$ is strong operator continuous on norm bounded subsets of $G_{\mathcal{A}}$.*

Proof. Let $B \subset G_{\mathcal{A}}$ be a norm bounded set and suppose that $\mathcal{A}$ is finite. Since B is metrizable in the strong operator topology, we can deal with sequences instead of nets. Let $g_n, g \in B$ such that g_n converges strongly to g. Then $g_n^* g_n$ converges strongly to $g^* g$, because $\mathcal{A}$ is finite and the sequence is bounded. By the strong continuity of the functional calculus it follows that $(g_n^* g_n)^{-1/2}$ converges strongly to $(g^* g)^{-1/2}$. Then one has strong convergence of the unitary parts of the polar decompositions,

$$u_n = g_n (g_n^* g_n)^{-1/2} \overset{\text{strongly}}{\rightarrow} u = g(g^* g)^{-1/2},$$

(here we use that the *sequence* $(g_n^* g_n)^{-1/2}$, being strong convergent, is norm bounded). Therefore u_n^* converges strongly to u^*, and

$$g_n^{-1} = (g_n^* g_n)^{-1/2} u_n^* \overset{\text{strongly}}{\rightarrow} g^{-1} = (g^* g)^{-1/2} u^*.$$

□

As a consequence of this lemma, one obtains that the canonical decomposition φ is continuous when restricted to $V_n^C(\mathcal{A})$, if $\mathcal{A}$ is finite. This is clear by the formula given for φ in the first section, in terms of products, $*$ operation and inversion. Also it can be seen that the boundedness restriction is necessary. On the other hand, always under the assumption that $\mathcal{A}$ is finite, φ is strongly continuous in $V_n^\perp(\mathcal{A})$, because it consists of elements with norm 1

To follow the same argument as in the previous section, this time with the strong topology, we need a result stating the existence of strongly continuous local cross sections for the set of systems of projections $P_n(\mathcal{A})$. In full generality ($\mathcal{A}$ finite) we do not know if this holds. However one can prove the existence of *global* strongly continuous cross sections for a special class of finite II_1 factors. Let $\mathcal{M}$ be a II_1 factor such that when tensored with $B(H)$ (H separable) admits a one parameter automorphism group θ_t which scales the trace τ of $\mathcal{M} \otimes B(H)$, i.e. $\tau \circ \theta_t = e^{-t}\tau$

For these factors S. Popa and M. Takesaki proved [12], among other results, that the unitary group $U_\mathcal{M}$ is strongly contractible and admits what E. Michael [11] calls a *geodesic structure.* In this setting, one can use Michael's continuous selection principle [11]: if $X \to X/Y$ is a quotient map, where X is a complete metric space, and Y admits a geodesic structure, then the quotient map admits a continuous global cross section. We shall apply these results to obtain a global cross section for the unitary orbits of elements $\vec{q} \in P_n(\mathcal{A})$. Let $P_n^0(\mathcal{A})$ be as before, the set of systems of projections where the projections are pairwise equivalent. Note that $P_n^0(\mathcal{A})$ is connected in norm, and therefore also in the strong operator topology.

Proposition 5.2 *Let $\mathcal{A}$ be finite. The map*

$$\pi_{\vec{p}} : U_\mathcal{A} \to P_n^0(\mathcal{A}), \quad \pi_{\vec{p}}(u) = (up_1u^*, ..., up_nu^*)$$

is open, when both sets are considered with the strong operator topology.

In order to prove this we need the following elementary result.

Lemma 5.3 *Let $\mathcal{A} \subset B(H)$ be a finite von Neumann algebra, and let $a_n \in \mathcal{A}$ such that $\|a_n\| \leq 1$ and $a_n^*a_n$ tends to 1 in the strong operator topology. Then there exist unitaries u_n in $\mathcal{A}$ such that $u_n - a_n$ converges strongly to zero.*

Proof. Consider the polar decomposition $a_n = u_n|a_n|$, where u_n can be chosen unitaries because $\mathcal{A}$ is finite. Note that $|a_n| \to 1$ strongly. Indeed, since $\|a_n\| \leq 1$, $a_n^*a_n \leq (a_n^*a_n)^{1/2}$. Therefore, for any unit vector $\xi \in H$, $1 \geq (|a_n|\xi, \xi) \geq (a_n^*a_n\xi, \xi) \to 1$. Therefore

$$\|(a_n - u_n)\xi\|^2 = \|u_n(|a_n| - 1)\xi\|^2 \leq \||a_n|\xi - \xi\|^2 = 1 + (a_n^*a_n\xi, \xi) - 2(|a_n|\xi, \xi),$$

which tends to zero. □

Proof (of the proposition). Let $u_k(p_1, ..., p_n)u_k^*$ be a sequence in $P_n^0(\mathcal{A})$ converging strongly to $(p_1, ..., p_n)$, i.e. $u_kp_iu_k^* \to p_i$ strongly to p_i for $i = 1, ..., n$. This implies that $(p_iu_kp_i)(p_iu_k^*p_i) \to p_i$. By the above lemma, applied in the finite von Neumann algebra $p_i\mathcal{A}p_i$ for each $i = 1, ..., n$, there exist unitaries $w_{k,i}$ in $p_i\mathcal{A}p_i$ such that $p_iu_kp_i - w_{k,i} \to 0$ strongly. Let $w_k = \sum_{i=1}^n w_{k,i}$. Then w_k is a unitary in $\mathcal{A}$ which commutes with $(p_1, ..., p_n)$. Then

$$u_kw_k^* = u_k(w_k^* - \sum_{i=1}^n p_iu_k^*p_i) + u_k\sum_{i=1}^n p_iu_k^*p_i \stackrel{strongly}{\to} 1.$$

Indeed, the first summand converges to 0, for each $\xi \in H$, $\|u_k(w_k^* - \sum_{i=1}^n p_iu_k^*p_i)\xi\| \leq \sum_{i=1}^n \|(w_{k,i} - p_iu_k^*p_i)\xi\|$ and each one of these terms tend to zero. The other summand $u_k\sum_{i=1}^n p_iu_k^*p_i = \sum_{i=1}^n (u_kp_iu_k^*)p_i \stackrel{strongly}{\to} \sum_{i=1}^n p_i = 1$.

If $\pi_{\vec{p}}(u_k) = u_k(p_1, ..., p_n)u_k^* \to \pi_{\vec{p}}(u) = u(p_1, ..., p_n)u^*$ strongly in $P_n^0(\mathcal{A})$, then $u^*u_k(p_1, ..., p_n)(u^*u_k)^*$ converges strongly to $(p_1, ..., p_n)$. By the computation above, there exist unitaries w_k commuting with $(p_1, ..., p_n)$, such that $u^*u_kw_k^* \to 1$ strongly, i.e $u_kw_k^* \to u$ strongly. Since w_k commutes with $\vec{p}$, $\pi_{\vec{p}}(u_kw_k^*) = \pi_{\vec{p}}(u_k)$. Therefore $\pi_{\vec{p}}$ is open. □

Suppose now that $\mathcal{A}$ is a II_1 factor which when tensored with $B(H)$ admits a one parameter group of automorphisms scaling the trace. Then one has the following

Theorem 5.4 *If $\mathcal{A}$ is a II_1 factor as above, the map*

$$\pi_{\vec{p}} : U_{\mathcal{A}} \to P_n^0(\mathcal{A}), \quad \pi_{\vec{p}}(u) = (up_1u^*, ..., up_nu^*)$$

is a trivial bundle in the strong operator topology.

Proof. Note that if $\mathcal{A}$ is finite, $U_{\mathcal{A}}$ is a complete metrizable topological group in the strong operator topology. We will show that $\pi_{\vec{p}}$ has a continuous global cross section. By the proposition above, $\pi_{\vec{p}}$ induces the homeomorphism

$$P_n^0(\mathcal{A}) \cong U_{\mathcal{A}}/U_{\mathcal{B}}$$

where $\mathcal{B} = \{p_1, ..., p_n\}' \cap \mathcal{A}$, and $\pi_{\vec{p}}$ is equivalent to the quotient map

$$\pi : U_{\mathcal{A}} \to U_{\mathcal{A}}/U_{\mathcal{B}}.$$

It suffices to show that this map has a continuous global cross section. Here we can apply the result of Popa and Takesaki [12] (based on Michael's theory of continuous selections [11]), because the fibre $U_{\mathcal{B}}$ has a geodesic structure. Indeed, $\mathcal{B} \cong p_1\mathcal{A}p_1 \oplus \ldots \oplus p_n\mathcal{A}p_n$, and each $p_i\mathcal{A}p_i$ is a factor which is *-isomorphic to $\mathcal{A}$, because the trace sclaing property implies $\mathcal{A} \cong p\mathcal{A}p$ for any non trivial projection $p \in \mathcal{A}$ (see [10], chapter 13). Then $U_{\mathcal{B}} \cong U_{p_1\mathcal{A}p_1} \times \ldots \times U_{p_n\mathcal{A}p_n}$ has a geodesic structure. Therefore π has a continuous global cross section. □

Let $a \in V_n(\mathcal{A})$ as before. Note that the map

$$\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A}), \quad \pi_a(g) = gag{-1}$$

is strongly continuous when restricted to norm bounded subsets of $G_{\mathcal{A}}$.

Proposition 5.5 *The map*

$$\pi_a : G_{\mathcal{A}} \to V_n(\mathcal{A}), \quad \pi_a(g) = gag{-1}$$

has a global cross section which is continuous in the strong operator topology on the norm bounded subsets $V_n^C(\mathcal{A})$ of $V_n(\mathcal{A})$.

Proof. Denote by $\Omega : P_n^0(\mathcal{A}) \to U_{\mathcal{A}}$ a cross section for $\pi_{\vec{p}}$. Recall the map Λ of 3.1

$$\Lambda(t) = \sigma_{\vec{p}}(\varphi(t))\mu(\sigma_{\vec{p}}(\varphi(t))^*t\sigma_{\vec{p}}(\varphi(t)))$$

and modify it by replacing the local cross section $\sigma_{\vec{p}}$ (of $\pi_{\vec{p}}$) by the global cross section Ω, i.e.

$$\Delta(t) = \Omega_{\vec{p}}(\varphi(t))\mu(\Omega_{\vec{p}}(\varphi(t))^*t\Omega_{\vec{p}}(\varphi(t))).$$

This map is strongly continuous on norm bounded subsets of $V_n(\mathcal{A})$. □

By the same argument as in 2.5 of the previous section, if $t \in V_n^{\perp}(\mathcal{A})$, then the global cross section Δ takes values in the unitary group $U_{\mathcal{A}}$. In this case the cross section is continuous in the whole $V_n^{\perp}(\mathcal{A})$. One has the following result

Proposition 5.6 *If $\mathcal{A}$ is a II_1 factor as above, then the map*

$$\pi_a : U_{\mathcal{A}} \to V_n^{\perp}(\mathcal{A}), \quad \pi_a(u) = uau^*$$

is a trivial bundle with fibre equal to the group of unitary operators which commute with a.

Corollary 5.7 *If $\mathcal{A}$ is a II_1 factor as above, then $P_n^0(\mathcal{A})$ and $V_n^{\perp}(\mathcal{A})$ in the strong operator topology, have trivial homotopy groups of all orders.*

Proof. Consider the fibrations $\pi_{\vec{p}}$ and π_a above. The total space and the fibres of both fibrations are contractible in the strong operator topology [12]. In the case of $\pi_{\vec{p}}$, the fibre is homeomorphic to $(U_{p\mathcal{A}p})^n \simeq (U_{\mathcal{A}})^n$. In the case of π_a, it is $U_{p\mathcal{A}p} \simeq U_{\mathcal{A}}$. $\square$

We return to the case of a general II_1 von Neumann algebra $\mathcal{A}$. Let t be an arbitrary (not necesarilly very nice Jordan) nilpotent of order n. We shall establish that the canonical decomposition φ is strongly continuous on this set on the unitary orbit of t. Since we do not have the formula of section 1 to compute the projections onto the kernels, we need the following result.

Proposition 5.8 *Let $\mathcal{A}$ be a finite algebra and P the set of projections of $\mathcal{A}$. For a fixed $p \in P$ denote by K_p the set of elements of $\mathcal{A}$ whose kernel projections are equivalent to p. Then the map*

$$k : K_p \to P, k(a) = P_{\ker a}$$

is continuous on norm bounded subsets of K_p, when both K_p and P are considered in the strong operator topology.

Proof. The proof is based on a result in [1], which states that the map which assigns to a positive normal functional its support projection is continuous when restricted to the set of positive functionals with equivalent supports, regarded with the norm topology, to the set P in the strong operator topology. Fix a faithful tracial state τ in $\mathcal{A}$. Suppose that b_n is a sequence (bounded in norm) in K_p which converges strongly to b in K_p. Let ψ_n, ψ be the positive normal functionals of $\mathcal{A}$ given by $\psi_n(x) = \tau(b_n^* x b_n)$ and $\psi(x) = \tau(b^* x b)$. Clearly the support of ψ_n is the projection onto the kernel of b_n, and the support of ψ is the projection onto the kernel of b, which are are equivalent projections by hypothesis. Note that $\|\psi_n - \psi\|$ tends to zero. Indeed,

$$|\psi_n(x) - \psi(x)| = |\tau(x(b_n^* b_n - b^* b))| \leq \tau(x^* x)^{1/2} \tau((b_n^* b_n - b^* b)^2)^{1/2} \leq \|x\| \|b_n^* b_n - b^* b\|_2$$

where $\| \ \|_2$ denotes the L^2-norm induced by τ. Since b_n are uniformly bounded in norm, and $\mathcal{A}$ is finite, it follows that $b_n^* b_n \to b^* b$ strongly, and therefore in the norm $\| \ \|_2$ as well. Then $\psi_n \to \psi$ in norm , and by the continuity result cited above, the supports converge strongly, i.e. $P_{\ker b_n} \to P_{\ker b}$ strongly. $\square$

Corollary 5.9 *Let $\mathcal{A}$ be a finite von Neumann algebra and $t \in N_n(\mathcal{A})$. The canonical decomposition φ restricted to the unitary orbit of t is continuous in the strong operator topology.*

Proof. For any $1 \leq k \leq n-1$, the map $utu^* \mapsto P_{\ker(utu^*)^k}$ is continuous in the strong operator topology by the result above. Indeed, since $(utu^*)^k = ut^k u^*$, the elements $(utu^*)^k$ have the same norm and equivalent kernels. Then it is clear that φ is continuous when restricted to this set. $\square$

The canonical decomposition restricted to the *similarity* orbit of an arbitrary $t \in N_n(\mathcal{A})$ is also continuous, but only on bounded subsets of the similarity orbit. The proposition above applies because if t is similar to t', then they have equivalent kernel projections.

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