GROUP CONDITIONAL EXPECTATIONS OF FINITE INDEX[1]

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with finite dimensional center, G a subgroup of the group of automorphisms of $\mathcal{M}$ such that $\mathcal{M}$ is G-finite and $E : \mathcal{M} \to \mathcal{M}^G = \{x \in \mathcal{M} : g(x) = x \text{ for all } g \in G\}$ a faithful normal conditional expectation. Then E has finite index if and only if the group G has compact closure in $B(\mathcal{M})$. The same result and its natural dinamical systems versions are proved for simple C*-algebras.

1. Introduction.

Let $\mathcal{M} \subset L(H)$ be a von Neumann algebra acting on the Hilbert space H. Let $\mathcal{N} \subset \mathcal{M}$ a subalgebra, and E a conditional expectation $E : \mathcal{M} \to \mathcal{N}$. After the introduction of Jones paper [J] several notions of index for conditional expectations between von Neumann algebras appeared ([PP], [K],[W]). A remarkable survey on the subject is the book of S. Popa [Po] (see also [BDH] and [W]).

When studying homogeneous reductive spaces of infinite dimension (modelled in C*-algebras), conditional expectations become an essential feature (see [MR]). For example, if the reductive structure is given by a conditional expectation of finite index, the homoge- neous space can be naturally imbedded as a submanifold of the grassmannians of a given C*-algebra (see [AS] or [ALRS]). On the other hand, a conditional expectation onto the centralizer of a group action can be used to describe the differential geometry of a suitable orbit of the action (see for example the case of unitary representations treated in [M], [MR] and [ALRS]).

(1.1) Definition. A conditional expectation $E : \mathcal{M} \to \mathcal{N}$ is said to be of finite index if there exists a positive scalar $K > 0$ such that the mapping $KE - Id_{\mathcal{M}}$ is

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positive. Denote by $Ind_w(E)$ (called the *probabilistic* index in [Po]) the infimum of all such K. If there is no such K, put $Ind_w(E) = \infty$.

In [Po] Popa proved that if there exists such a constant K then there exists another positive constant L such that $LE - Id_{\mathcal{M}}$ is completely positive. Frank and Kirchberg extended this result for conditional expectations between C*-algebras. These theorems unify the two notions of finiteness for the index considered in [BDH]. In particular, the condition $Ind_w(E) < \infty$ of (1.1) is equivalent to the existence of a a $\mathcal{N}$-orthonormal basis $\{m_i\}_{i\in I}$ of the $\mathcal{N}$-right Hilbert module $\mathcal{M}$ (see (2.4)) such that $\sum_i m_i m_i^*$ is weakly convergent. Moreover this limit is the same for every $\mathcal{N}$-orthonormal basis and is denoted by $Ind(E)$.

Let G be a subgroup of the group $Aut(\mathcal{M})$ of automorphisms of $\mathcal{M}$. Let us recall the following definition, introduced by Kovacs and Szücz in [KS]:

(1.2) Definition. $\mathcal{M}$ is G-finite if for each positive element $x \in \mathcal{M}^+$ there exists a normal positive functional $\varphi \in \mathcal{M}_*$ such that
- $\varphi \circ g = \varphi$ for all $g \in G$.
- $\varphi(x) > 0$.

Kovacs and Szücz' [KS] main result states that this condition is equivalent to the existence of a necesarily unique, faithful, normal and G-invariant conditional expectation $E_G : \mathcal{M} \to \mathcal{M}^G$ (G-invariance means that $E_G \circ g = E_G$ for all $g \in G$).

In this paper we study the finite index condition for these expectations E_G. Note that if $\mathcal{M}$ is G-finite, due to the uniqueness condition, the index of E_G can be regarded as the index of the inclusion $\mathcal{M}^G \subset \mathcal{M}$. Theoren 4.5 states that if $\mathcal{M}$ has finite dimensional center and is G-finite with E_G of finite index, then the closure of G is compact in the norm topology of $B(\mathcal{M})$. The converse of these statement is also true, even in the case when $\mathcal{M}$ has infinite dimensional center. As a corollary, applying standard arguments ([BDH]) we obtain the same result for arbitrary normal (not necesarily G-invariant) expectations.

In section 2. we state some basic facts concerning finite index expectations on von Neuman algebras. These will be used in the next sections.

In section 3. we prove our statement for groups of inner automorphisms. In this case a stronger result holds: the mere existence of a finite index expectation from $\mathcal{M}$ onto $\mathcal{M}^G$ implies that $\mathcal{M}$ is G-finite and that also E_G has finite index.

In section 4. the main results are proven, by reducing the outer case to the inner case in the Jones extension $\mathcal{M}_1$ of $\mathcal{M}$.

In section 5. we prove the analogous of (4.5) for simple C*-algebras.

2. Preliminaries.

In this section we state several well known results which will be used in the next sections.

(2.1) ([BDH] 3.19 and [Po] 1.1.2 (iv)) Let $E : \mathcal{M} \to \mathcal{N}$ a conditional expectation with $Ind_w(E) < \infty$. The following are equivalent
- $\dim \mathcal{Z}(\mathcal{M}) < \infty$
- $\dim \mathcal{Z}(\mathcal{N}) < \infty$
- $\dim(\mathcal{N}' \cap \mathcal{M}) < \infty$

(2.2) Let $E : \mathcal{M} \to \mathcal{N}$ a conditional expectation with $Ind_w(E) < \infty$.
1) ([BDH] 3.17) If $\mathcal{N}$ is finite dimensional then $\mathcal{M}$ is finite dimensional.
2) Let $\mathcal{A} \subset \mathcal{M}$ a von Neumann algebra. Suppose that $\mathcal{N} = \mathcal{A}' \cap \mathcal{M}$ and $\dim \mathcal{Z}(\mathcal{M}) < \infty$, then $\dim \mathcal{A} < \infty$.

Since $\dim \mathcal{Z}(\mathcal{M}) < \infty$ and $Ind_w(E) < \infty$, by 2.1 we have that $(\mathcal{A}' \cap\ M)' \cap \mathcal{M}$ is also finite dimensional. But $\mathcal{A} \subset (\mathcal{A}' \cap\ M)' \cap \mathcal{M}$, and therefore 2) holds.

(2.3) Proposition. *Let $\mathcal{A} \subset \mathcal{M}$ be von Neumann algebras and $E : \mathcal{M} \to \mathcal{A}' \cap \mathcal{M}$ a conditional expectation. If $\dim \mathcal{A} < \infty$, then $Ind_w(E) < \infty$.*

Proof. We include a proof of this natural result because it doesn't seems to be a direct consequence of known (to us) published results and because it is elemental and we shall need it later. Let $p_1, ..., p_N$ be (orthogonal) minimal projections of $\mathcal{Z}(\mathcal{A})$ which sum 1. Let $F : \mathcal{M} \to \mathcal{Z}(\mathcal{A})' \cap \mathcal{M}$ the conditional expectation defined by $F(x) = \sum_{i=1}^N p_i x p_i$. It is fairly known that $Ind_w(F) \leq N$. Since $p_i \in \mathcal{Z}(\mathcal{A})' \cap \mathcal{M}$ then $E \circ F = E$. Therefore

$$Ind_w(E) \leq Ind_w(E_0)\ Ind_w(F) = N\, Ind_w(E_0),$$

where $E_0 = E|_{\mathcal{Z}(\mathcal{A})' \cap \mathcal{M}}$. Note that $E_0 = \sum_{i=1}^N E_i$ where

$$E_i : p_i \mathcal{M} p_i \to (p_i \mathcal{A} p_i)' \cap p_i \mathcal{M} p_i$$

is given by $E_i(p_ixp_i) = p_iE(x)$. Clearly the maps E_i are faithful conditional expectations and

$$Ind_w(E_0) = \sup_i Ind_w(E_i).$$

Therefore it suffices to show that each E_i has finite index. In other words, we can suppose that $\mathcal{A}$ is a (finite dimensional) factor. Then the restriction $E|_{\mathcal{A}}$ is a faithful state φ of $\mathcal{A}$. On the other hand we have the natural isomorphism

$$\mathcal{M} \cong (\mathcal{A}' \cap \mathcal{M}) \otimes \mathcal{A}.$$

Since $E|_{\mathcal{A}'\cap\mathcal{M}} = Id$ it follows that (via the natural isomorphism) $E = Id \otimes \varphi$. Since φ is faithful there exists a positive and invertible element $h \in \mathcal{A}$ such that $\varphi(x) = \tau(hx)$ where τ is the normalized trace of $\mathcal{A}$. Note that if $E_\tau = Id \otimes \tau$ then $Ind(E_\tau) = N$, where $N^2 = \dim \mathcal{A}$. Therefore, if $x \in \mathcal{M}^+$,

$$E(x) = E_\tau((1 \otimes h)x) \geq \|h^{-1}\|^{-1} E_\tau(x) \geq 1/N \|h^{-1}\|^{-1} x\,.$$

This shows that E has finite index.

(2.4) Let $E : \mathcal{M} \to \mathcal{N}$ a faithful normal conditional expectation. Let us recall Jones' basic construction. Fix a faithful normal state φ in $\mathcal{N}$. Denote also by φ the faithful and normal state on $\mathcal{M}$ given by $\varphi \circ E$. Consider $\mathcal{M}$ acting on the Hilbert space $L^2(\mathcal{M}, \varphi)$ of the GNS construction of φ. Denote by ξ the ciclic vector (=1 of $\mathcal{M}$ regarded as a vector in $L^2(\mathcal{M}, \varphi)$). Let $J = J_\varphi$ be the modular antiunitary operator such that $JMJ = M'$.

Denote by e the Jones' projection of E, that is, the orthogonal projection from $L^2(\mathcal{M}, \varphi)$ onto $L^2(\mathcal{N}, \varphi)$. Put $\mathcal{M}_1$ the von Neumann algebra generated by $\mathcal{M}$ and e.

Among the well known properties of e recall ([Po], [Ko], [Jo]) that

- $exe = E(x)e$ for all $x \in \mathcal{M}$.
- $\mathcal{M} \cap \{e\}' = \mathcal{N}$.
- Finite sums of the form $\sum_i x_iex_i'$ with $x_i, x_i' \in \mathcal{M}$ are weakly dense in $\mathcal{M}_1$.
- $\mathcal{M}_1 = J\mathcal{N}'J$.
- $JeJ = e$.

– There exists a unique operator valued weight E^{-1} of $\mathcal{M}_1$ to $\mathcal{M}$ defined on the finite sums of the form $\sum_i x_i e x'_i$ by

$$E^{-1}(\sum_i x_i e x'_i) = \sum_i x_i x'_i$$

– A family $\{m_i\}_i \subset \mathcal{M}$ is called an $\mathcal{N}-$*orthonormal basis* of $\mathcal{M}$ if $\sum_i m_i e m_i^* = 1$ and $E(m_i^* m_j) = \delta_{ij} p_i \in \mathcal{P}(\mathcal{N})$. In this case it verifies that for $x \in \mathcal{M}$,

$$x = \sum_i m_i E(m_i x),$$

where the sum converges in the σ topology (defined by the $\varphi \in \mathcal{M}_*^+$ such that $\varphi = \varphi \circ E$).

Let us summarize the facts equivalent to the finite index condition for E (see [Po], [BDH] or [FK]).

(2.5) Theorem. *([Po, Th. 1.1.6] and [BDH]) With the above notations, the following are equivalent:*

1) $Ind_w(E) < \infty$.

2) There exists $K > 0$ *such that* $K\|E(a)\| \geq \|a\|$ *for all* $a \in \mathcal{M}^+$.

3) E^{-1} *is everywhere defined, i.e.* $E^{-1}(1)$ *is bounded.*

4) There exist a $\mathcal{N}$*-orthonormal basis* $\{m_i\}_i$ *of* $\mathcal{M}$ *such that* $\sum_i m_i m_i^*$ *is a bounded operator.*

In this case, for each $\mathcal{N}$*-orthonormal basis* $\{m_i\}_i$*, one has*

$$Ind(E) = \sum_i m_i m_i^* = E^{-1}(1) \in \mathcal{Z}(\mathcal{M})$$

which is invertible. Moreover the map $E_1 = Ind(E)^{-1}E^{-1} : \mathcal{M}_1 \to \mathcal{M}$ *is a finite index conditional expectation such that* $E_1(e) = Ind(E)^{-1}$*. Also* $Ind_w(E)$ *is the best constant for 3).*

3. Groups of unitaries.

Let $G \subset \mathcal{U}_{\mathcal{M}}$ (=unitary group of $\mathcal{M}$) be a group such that there exists a faithful conditional expectation $E : \mathcal{M} \to \mathcal{M}^G = \{x \in \mathcal{M} : gxg^* = x,\ g \in G\}$. Denote by $\mathcal{A}_G = G''$. Note that $\mathcal{M}^G = \mathcal{A}'_G \cap \mathcal{M}$.

(3.1) Theorem. *Let G be a group of unitary elements in a factor $\mathcal{M}$ and $E : \mathcal{M} \to \mathcal{M}^G$ a faithful conditional expectation. The following facts are equivalent:*
a) $E : \mathcal{M} \to \mathcal{M}^G$ has $finite$ index.
b) $\mathcal{A}_G$ is finite dimensional.
c) The norm closure of G is compact in the $\| \|$-topology (of $\mathcal{M}$).
In particular, if these conditions are fulfilled, E is normal.

Proof . b) $\Rightarrow$ c) is trivial.
a)$\Rightarrow$ b) follows from (2.2).
c)$\Rightarrow$ a): Let m be the Haar measure of $\bar{G}^{\| \|}$ and put

$$E_G : \mathcal{M} \to \mathcal{M}^G\,, \quad E_G(x) = \int_{\bar{G}^{\| \|}} gxg^* dm(g).$$

Clearly E_G is a faithful and normal conditional expectation. In [ALRS] (Prop 5.3) it was shown that when the group is norm-compact, then the index of E_G is finite, even for C*-algebras. Therefore (using a) $\Rightarrow$ b)) we can assume that $\mathcal{A}_G$ is finite dimensional. This implies (by (2.3) and (2.1)) that our original expectation E has finite index.

The following Corollary provides a converse of Prop. 5.3 of [ALRS] for the case of a factor.

(3.4) Corollary *Let G be a compact group, $\mathcal{M}$ a factor and $\pi : G \to \mathcal{U}_{\mathcal{M}}$ a SOT continuous representation. Let m be the Haar measure of G and $E_G = \int_G Ad_{\pi(g)} dm(g)$ the canonical conditional expectation $E_G : \mathcal{M} \to \mathcal{M}^G$. The following are equivalent:*
i) π is norm continuous.
ii) E_G has finite index.
iii) $\dim(\pi(G)'') < \infty$.

(3.5) Remark. Theorem 3.1 can be extended, with almost the same proof, to hold for algebras $\mathcal{M}$ with $\dim(\mathcal{Z}(\mathcal{M})) < \infty$.

(3.6) Remark. Theorem 3.1 provides a proof for the following elementary fact: Let G be a group of unitaries of a factor $\mathcal{M}$, then the norm closure of the group is compact in the norm topology if and only if the von Neumann algebra G'' generated by G is finite dimensional.

(3.7) Corollary *Let G be a group of unitaries of a factor $\mathcal{M}$ and suppose that there exists a conditional expectation $F : \mathcal{M} \to \mathcal{M}^G$ with finite index. Then $\mathcal{M}$ is $Inn(G)$-finite, where $Inn(G)$ denotes the group of inner automorphisms induced by G.*

Proof . By (3.1) $\bar{G}^{\| \|}$ is compact in the norm topology. Let m denote its Haar measure. The canonical conditional expectation

$$E_G : \mathcal{M} \to \mathcal{M}^G\,, \quad E_G(x) = \int_G gxg^* dm(g)$$

is G-invariant, faithful and normal.

4. Arbitrary groups of automorphisms.

From now on, unless otherwise stated, $G \subset Aut(\mathcal{M})$ is an arbitrary group of automorphisms such that $\mathcal{M}$ is G-finite. Denote by E_G the canonical (faithful and normal) conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^G$.

Recall Jones' basic construction (2.4). Fix a faithful normal state φ in $\mathcal{M}^G$. Denote also by φ the faithful and normal state on $\mathcal{M}$ given by $\varphi \circ E_G$. Note that φ is G-invariant ($\varphi \circ g = \varphi$ for all $g \in G$). Consider $\mathcal{M}$ acting on the Hilbert space $L^2(\mathcal{M}, \varphi)$ of the GNS construction of φ. Let ξ be the ciclic vector and $J = J_\varphi$ the modular antiunitary operator such that $JMJ = M'$.

(4.1) Remark. Every $g \in G$ gives rise to a unitary operator U_g in $L^2(\mathcal{M}, \varphi)$ defined by $U_g(x\xi) = g(x)\xi$, for $x \in \mathcal{M}$. This follows from the fact that $\varphi \circ g = \varphi$, which can be read as $< U_g x\xi, U_g y\xi >_\varphi = < x\xi, y\xi >_\varphi$ for all $x, y \in \mathcal{M}$. Clearly $\{U_g : g \in G\}$ is a group of unitaries. Moreover, the unitaries U_g satisfy

– For $x \in \mathcal{M}$, we have $U_g x U_g^* = g(x)$.
– $JU_gJ = U_g$ for all $g \in G$.
The second statement follows from the fact that $g \in G$ are *-morphisms.

Denote by e the Jones' projection of E_G. Put $\mathcal{M}_1$ the von Neumann algebra generated by $\mathcal{M}$ and e as in (2.4).
Let us also state two elementary facts regarding the operators U_g, $g \in G$ and e.

(4.2) Lemma.
– $\big(\mathcal{M} \cup \{U_g : g \in G\}\big)'' = \mathcal{M}_1$.
– $U_g e U_g^* = e$ for all $g \in G$.

Proof . Since $\mathcal{M} \cap \{U_g : g \in G\}' = \mathcal{M}^G$, using the properties of J cited above

$$\begin{aligned}\mathcal{M}_1 &= J(\mathcal{M}^G)'J = J\big(\mathcal{M} \cap \{U_g : g \in G\}'\big)'J = \big(\mathcal{M}' \cap \{JU_gJ : g \in G\}'\big)' \\ &= \big(\mathcal{M} \cup \{U_g : g \in G\}\big)''.\end{aligned}$$

The second statement follows from the fact that G acts trivially on $\mathcal{M}^G$ and therefore the unitaries U_g are the identity on the subspace of $L^2(\mathcal{M}, \varphi)$ spanned by $\mathcal{M}^G.\xi$ which is precisely the range of e.

(4.3) Remark. Suppose that E_G has finite index. By (2.5) there exists a faithful and normal conditional expectation $E_1 : \mathcal{M}_1 \to \mathcal{M}$ with finite index. Also $Ind(E) = E_1(e)^{-1} = \sum_{i \in I} m_i m_i^* \in Z(\mathcal{M})$ for any $\{m_i\}_{i \in I}$ E_G-orthonormal basis of the $\mathcal{M}^G$-right Hilbert module $\mathcal{M}$.

Denote by F the conditional expectation $E_G \circ E_1 : \mathcal{M}_1 \to \mathcal{M}^G$. Note that F has finite index. Denote by G_1 the group of inner automorphisms of $\mathcal{M}_1$ given by the group of unitaries $\{U_g : g \in G\}$.

(4.4) Proposition. *With the notations above, if the index of E_G is finite, then $\mathcal{M}_1$ is G_1- finite. Moreover, if E_{G_1} is the unique G_1-invariant conditional expectation $E_{G_1} : \mathcal{M}_1 \to \mathcal{M}_1^{G_1} = \mathcal{M}_1 \cap \{U_g : g \in G\}'$, then $F \circ E_{G_1} = F$.*

Proof . Since E_G has finite index, by 4.3 there exists a faithful and normal conditional expectation $E_1 : \mathcal{M}_1 \to \mathcal{M}$ such that $Ind(E) = E_1(e)^{-1} = \sum_{i \in I} m_i m_i^* \in Z(\mathcal{M})$ for any $\{m_i\}_{i \in I}$ E_G-ortonormal basis of the $\mathcal{M}^G$-right Hilbert module $\mathcal{M}$.

Put $\psi = \varphi \circ E_1$. Clearly ψ is a faithful normal state of $\mathcal{M}_1$. Let us see that it is G_1-invariant. The elements of the form $\sum_i x_i e x'_i$ with $x_i, x'_i \in \mathcal{M}$ are weakly dense in $\mathcal{M}_1$, therefore it suffices to prove that $\psi(U_g xey U_g^*) = \psi(xey)$ for all $x, y \in \mathcal{M}$ and $g \in G$.

$$\begin{aligned}\psi(U_g xey U_g^*) &= \varphi(E_1(U_g x U_g^* U_g e U_g^* U_g y U_g^*)) \\ &= \varphi(g(x) E_1(e) g(y)) \\ &= \varphi(x g^{-1}(E_1(e)) y).\end{aligned}$$

The statement follows if we prove that $g(E_1(e)) = E_1(e)$ for all $g \in G$. Let $\{m_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{M}$ as above. We claim that for any $g \in G$, $\{g(m_i)\}_{i \in I}$ is also an orthonormal basis:

- $E_G(g(m_i)^* g(m_j)) = E_G(g(m_i^* m_j)) = E_G(m_i^* m_j) = 0$ if $i \neq j$.
- For any $x \in \mathcal{M}$,

$$x = g(g^{-1}(x)) = g(\sum_{i \in I} E_G(m_i^* g^{-1}(x)) m_i = \sum_{i \in I} E_G(g(m_i)^* x) g(m_i).$$

Therefore

$$g(E_1(e)^{-1}) = g(\sum_{i \in I} m_i m_i^*) = \sum_{i \in I} g(m_i) g(m_i^*) = E_1(e)^{-1}$$

and our first claim is proven. By Theorem 1 of [KS], $E_{G_1}(x)$ is the unique point in $\mathcal{M}_1^{G_1} \cap \overline{co}^w \{U_g x U_g^* : g \in G\}$, where $\overline{co}^w$ denotes the weak closure of the convex hull. Therefore $\psi \circ F \circ E_{G_1} = \psi \circ E_{G_1} = \psi$. Since ψ is faithful and normal and also leaves F invariant, it follows that $F = F \circ E_{G_1}$.

Next we prove our main result

(4.5) Theorem. *Let $\mathcal{M}$ be a von Neumann algebra and G a group of automorphisms of $\mathcal{M}$. Then*

a) If $\mathcal{M}$ has finite dimensional center and is G-finite with E_G of finite index, then $\bar{G}^{\| \|}$ is compact in $B(\mathcal{M})$.

b) If $\bar{G}^{\| \|}$ is compact in $B(\mathcal{M})$ then $\mathcal{M}$ is G-finite and E_G is of finite index.

Proof . Let us first prove a):
By (4.4), $F \circ E_{G_1} = F$ and therefore $E_{G_1} : \mathcal{M}_1 \to \mathcal{M}_1^{G_1}$ has finite index. In order to apply (3.5) and (3.1) to the expectation E_{G_1}, we need to show that $\mathcal{M}_1$ has finite dimensional center. This follows from (2.4), since $\mathcal{M}_1 = J(\mathcal{M}^G)'J$ and $\mathcal{M}^G$ has finite dimensional center by (2.1). Therefore $\{U_g : g \in G\}$ has compact closure in $B(L^2(\mathcal{M}, \varphi))$ with the norm topology. The mapping from the unitary group of $L^2(\mathcal{M}, \varphi)$ to $B(\mathcal{M}, B(L^2(\mathcal{M}, \varphi)))$ which maps U to $Ad(U)|_{\mathcal{M}}$ is continuous. Hence our statement follows noting that this mapping takes $\{U_g : g \in G\}$ onto G.

Let us now prove b):
The proof follows the lines of Prop. 5.3 of [ALRS]. Let m be be the Haar measure of the compact group $\bar{G}^{\| \|} \subset Aut(\mathcal{M})$. Then $\mathcal{M}$ is G-finite and E_G is given by

$$\mathcal{M} \ni x \mapsto \int_{\bar{G}^{\| \|}} g(x) dm(g) \in \mathcal{M}^G$$

Let us show that it has finite index. Fix $\epsilon > 0$ and put $\mathcal{V} = \{g \in \bar{G}^{\| \|} : \|g - Id_{\mathcal{M}}\| < \epsilon\}$. Since $\mathcal{V}$ is open in $\bar{G}^{\| \|}$, $m(\mathcal{V}) = \delta > 0$. If $a \in \mathcal{M}^+$, take a cyclic representation π of $\mathcal{M}$ on H with cyclic vector η such that $< \pi(a)\eta, \eta >= \|a\|$. Then

$$\begin{aligned} \|E_G(a)\| &\geq < \pi(E_G(a))\eta, \eta >= \int_{\bar{G}^{\| \|}} < \pi(g(a))\eta, \eta > dm(g) \\ &\geq \int_{\mathcal{V}} < \pi(g(a))\eta, \eta > dm(g) \\ &= \int_{\mathcal{V}} < (\pi(g(a)) - \pi(a))\eta, \eta > dm(g) + \int_{\mathcal{V}} < \pi(a)\eta, \eta > dm(g) \\ &= \int_{\mathcal{V}} < (\pi(g(a)) - \pi(a))\eta, \eta > dm(g) + \delta\|a\|. \end{aligned}$$

Note that

$$| < \pi(g(a) - a)\eta, \eta > | \leq \|g(a) - a\| \leq \|a\| \|g - Id_{\mathcal{M}}\| < \epsilon\|a\|,$$

and therefore $\int_{\mathcal{V}} < \pi(g(a) - a)\eta, \eta > dm(g) \geq -\epsilon\delta\|a\|$. Hence we obtain that

$$\|E_G(a)\| \geq \delta(1 - \epsilon)\|a\|.$$

The proof is complete taking $\epsilon < 1$ and using (2.5).

(4.6) Corollary. *Let $\mathcal{M}$ be a von Neumann algebra with finite dimensional center, G a group of automorphisms of $\mathcal{M}$ such that $\mathcal{M}$ is G-finite and $E : \mathcal{M} \to \mathcal{M}^G$ a faithful normal conditional expectation. Then E has finite index if and only if $\bar{G}^{\|\ \|}$ is compact in $B(\mathcal{M})$.*

Proof. If E has finite index, then $(\mathcal{M}^G)' \cap \mathcal{M}$ is finite dimensional (2.1). Using (3.15) of [BDH], any other faithful normal conditional expectation has also finite index. In particular E_G, and the statement follows.

Conversely, if $\bar{G}^{\|\ \|}$ is compact in $B(\mathcal{M})$, then E_G has finite index. Therefore again $(\mathcal{M}^G)' \cap \mathcal{M}$ is finite dimensional, and by the same argument E must have finite index.

(4.7) Corollary. *Let $\mathcal{M}$ be a von Neumann algebra with finite dimensional center, G a locally compact group and $\alpha : G \to Aut(\mathcal{M})$ a von Neumann dinamical system.*

1) *There exists an α-invariant conditional expectation $E : \mathcal{M} \to \mathcal{M}^G$ with $Ind_w(E) < \infty$ if and only if the norm closure of $\alpha(G)$ in $\mathcal{B}(\mathcal{M})$ is compact.*
2) *If G is compact and $E_G : \mathcal{M} \to \mathcal{M}^G$ its natural α-invariant conditional expectation, then $Ind_w E_G < \infty$ if and only if α is norm continuous.*

Proof. 1) is clear from (4.5).

2) The conditional expectation E_G is normal because of the σ-additivity of the Haar measure of G. Then $\mathcal{M}$ is $\alpha(G)$-finite and the basic construction of the proof of (4.5) can be used. We deduce the existence of a unitary representation $g \mapsto U_{\alpha(g)}$. By Corollary (3.4), this map is norm continuous and then also α is. The converse is clear by the first part.

(4.8) Remark. If $\mathcal{M}$ has infinite dimensional center, property a) of (4.5) may not hold. Take $\mathcal{A}$ an infinite dimensional abelian algebra, put $\mathcal{M} = M_2(\mathcal{A})$ and $G = Inn(\mathcal{M})$. Then $\mathcal{M}^G = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathcal{A}\}$. The expectation E_G is given by

$$E_G(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix}.$$

Clearly E_G has finite index (index = 2) and G is closed but non compact.

If S is a set, denote by $\#S$ its cardinal.

(4.9) Corollary *Let σ be an automorphism of the von Neumann algebra $\mathcal{M}$ with finite dimensional center. Suppose that there exists a conditional expectation $E : \mathcal{M} \to \mathcal{M}^\sigma = \{x \in \mathcal{M} : \sigma(x) = x\}$ with $E \circ \sigma = E$. Then E has finite index if and only if there exists a least positive integer m and a unitary element $v \in \mathcal{M}$ such that $\sigma^m = Ad(v)$ and v has finite spectrum. If $\mathcal{M}$ is a factor, one has that*

$$\max\{m, \#(sp(v))\} \leq Ind_w(E).$$

Proof . By the previous result it is clear that there exists a postive integer m such that σ^m is an inner automorphism, say $Ad(v)$, $v \in \mathcal{M}$. Otherwise σ would be properly outer and the group $< \sigma >$ would be infinite and discrete in the norm topology.

Kovacs and Szücz [KS] showed that if G_0 is a subgroup of G and M is G-finite, then it is also G_0-finite and the expectation E_G factorizes through E_{G_0}. That is, $\mathcal{M}^G \subset \mathcal{M}^{G_0} \subset \mathcal{M}$ and $E_G = E_1 \circ E_{G_0}$, where $E_{G_0} : \mathcal{M} \to \mathcal{M}^{G_0}$ and $E_1 = E_G|_{\mathcal{M}^{G_0}} : \mathcal{M}^{G_0} \to \mathcal{M}^G$. Therefore $Ind_w(E_{G_0}) \leq Ind_w(E_G)$, and the first index is finite if the second is. In our case if we put G the group generated by σ and G_0 the group generated by σ^m, it follows that $E_{<\sigma^m>} : \mathcal{M} \to \mathcal{M}^{\sigma^m} = \mathcal{M} \cap \{v\}'$ has finite index. By 3.1 it follows that that von Neumann algebra generated by v is finite dimensional and therefore v has finite spectrum.

The converse statement follows from the submultiplicative property of the index (see [Jo]), i.e. with the notations above,

$$Ind_w(E_{G_0}) Ind_w(E_1) \geq Ind_w(E_G) \geq \max\{Ind_w(E_{G_0}),\ Ind_w(E_1)\}.$$

The inequality $\max\{m, \#(sp(v))\} \leq Ind_w(E)$ follows from the *max* inequality cited above, using that when $\mathcal{M}$ is a factor the index of $E_{Ad(v)} : \mathcal{M} \to \mathcal{M} \cap \{v\}'$ is the number of minimal spectral projections of v.

(4.10) Corollary *Let G be a group of* **outer** automorphisms of a von Neumann algebra $\mathcal{M}$ with finite dimensional center. Suppose that $\mathcal{M}$ is G-finite. Then the index of E_G is finite if and only if G is finite.

Proof . As in the previous result, recall that a set of outer automorphisms is discrete in the norm topology of $B(\mathcal{M})$.

This last result can also be derived from the fact that in the outer case, the algebra $\mathcal{M}_1$ of the basic construction is just the cross product of $\mathcal{M}$ by the discrete group G.

(4.11) Remark. If $\mathcal{M}$ has finite dimensional center and is G-finite with E_G of finite index, then the mapping

$$G \ni g \mapsto U_g \in B(L^2(\mathcal{M}, \varphi))$$

of 4.1 is a **norm continuous** unitary representation of G (3.4). Therefore, if $\mathcal{R}$ is a von Neumann algebra between $\mathcal{M}$ and $B(L^2(\mathcal{M}, \varphi))$ which is (globally) $Ad(U_g)$ invariant for all $g \in G$, then $\mathcal{R}$ is G finite and the index of the canonical G inavariant conditional expectation is finite.

5. Simple C*-algebras.

In [FK] Frank and Kirchberg proved the equivalence of the two notions of finite index (Definition (1.1)) for conditional expectations between general C*-algebras $\mathcal{B} \subset \mathcal{A}$. The basic idea in their proof is to extend the conditional expectation E to the atomic part of the enveloping von Neumann algebra $\mathcal{A}''$, preserving both finite index conditions and then reduce to the case of conditional expectations between atomic von Neumann algebras. A very symilar argument allows us to extend Theorem (4.5) for simple C*-algebras:

(5.1) Theorem. *Let $\mathcal{A}$ be a C*-algebra. Let G be a subgroup of $Aut(\mathcal{A})$ and $\mathcal{A}^G = \{a \in \mathcal{A} : \ g(a) = a \text{ for } g \in G\}$. Then*

a) *If $\mathcal{A}$ is* **simple** and there exists a G- invariant conditional expectation $E_G : \mathcal{A} \to \mathcal{A}^G$ such that $Ind_w(E_G) < \infty$, then the norm closure of G in $\mathcal{B}(\mathcal{A})$ is compact.

b) If the norm closure of G is compact, then there exists a G- invariant conditional expectation $E_G : \mathcal{A} \to \mathcal{A}^G$ such that $Ind_w(E_G) < \infty$.

Proof.

a) Take $\mathcal{A}''$ the enveloping von Neumann algebra of $\mathcal{A}$, i.e the weak closure of the image of $\mathcal{A}$ in its universal representation. It is well known (see [FK], [Pe] or [KR] for general results) that we can assume the following facts:

1) The weak closure of $\mathcal{A}^G$ in $\mathcal{A}''$ is its enveloping algebra $(\mathcal{A}^G)''$.
2) E_G extends to a normal conditional expectation $E'' : \mathcal{A}'' \to (\mathcal{A}^G)''$ with the same index as E_G.
3) The group G can be extended to a group G'' acting on $\mathcal{A}''$: each $g \in G$ extends to an automorphism g'' $(= g^{**})$ $\mathcal{A}''$ with $g''|_{\mathcal{A}} = g$.

Note that E'' is G''-invariant and $E''(\mathcal{A}'') = (\mathcal{A}^G)'' \subset (\mathcal{A}'')^{G''}$.

Claim: $\mathcal{A}''$ is G''- finite and the conditional expectation $E_{G''}$ of [KS] has finite index.

Indeed, given $0 \leq x \in \mathcal{A}''$, take a normal state φ on $(\mathcal{A}^G)''$ such that $\varphi(E''(x)) \neq 0$. It is clear that the state $\varphi \circ E''$ is G''-invariant, then $\mathcal{A}''$ is G''- finite. By [KS], for $y \in \mathcal{A}''$, $E_{G''}(y)$ is the unique point of $(\mathcal{A}'')^{G''}$ in the weak closure of the convex hull of $\{g''(y) : g'' \in G''\}$. Therefore it is easy to see that $E'' \circ E_{G''} = E''$ and this implies that $Ind(E_{G''}) \leq Ind(E'') < \infty$.

By [Po, 1.1.2], the atomic central projections of $\mathcal{A}''$ and $(\mathcal{A}'')^{G''}$ are the same, namely $P_a \in \mathcal{Z}(\mathcal{A}'') \cap \mathcal{Z}(\mathcal{A}'')^{G''}$. Recall from the general theory of C*-algebras that $P_a \neq 0$.

Denote by $\mathcal{M} = P_a\mathcal{A}''$ and $\mathcal{N} = P_a(\mathcal{A}'')^{G''}$. Clearly $E_{G''}$ and G'' are reduced by P_a. Denote by $G_a \subset Aut(\mathcal{M})$ the restriction of G'' to $\mathcal{M}$ and E_a the reduction of $E_{G''}$ by P_a. It is apparent that $\mathcal{N} = \mathcal{M}^{G_a}$ and that $E_a = E_{G_a}$ (the unique G_a-invarinat conditional expectation) has finite index. It is a standard fact that $\mathcal{A}$ is faithfully represented in $\mathcal{M}$.

In order to apply Th. (4.5) we need $\mathcal{M}$ to have finite dimensional center. We don't now if the atomic part of the enveloping von Neumann algebra of a simple C*-algebra does have finite dimensional center. But in any case the following argument holds:

Since $\mathcal{M}$ is atomic, $\mathcal{Z}(\mathcal{M})$ is also atomic. Let p be a minimal central projection of $\mathcal{M}$. It is easy to see, using $Ind(E_a) < \infty$, that the set $R = \{g(p) \ : \ g \in G_a\}$ is finite. Let $R = \{p_1, \ldots, p_n\}$ and P the projection given by $P = \sum_{i=1}^n p_i \in$

$\mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{M}^{G_a})$. Proceeding again as above, we can restrict all our objets to the reduced algebras $P\mathcal{M}^{G_a} \subset P\mathcal{M}$. Since $\mathcal{A}$ is simple, it remains faithfully represented in $P\mathcal{M}$. Note that $\dim \mathcal{Z}(P\mathcal{M}) = n < \infty$. Now it is licit to apply (4.5) and then one deduces that if $Ind_w(E_G) < \infty$, then G has compact norm closure. We have proved a).

b) The proof is similar to the proof of (4.5) b) and we omit it.

(5.2) Corollary. *Let $\mathcal{A}$ be a simple C^*-algebra, G a locally compact group and $\alpha : G \to Aut(\mathcal{A})$ a C^* dynamical system. Then there exists a α-invariant conditional expectation $E : \mathcal{A} \to \mathcal{A}^G$ with $Ind_w(E) < \infty$ if and only if the norm closure of $\alpha(G)$ in $\mathcal{B}(\mathcal{A})$ is compact.*

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