Geometry of unitary orbits of oblique projections*

Esteban Andruchow, Gustavo Corach and Alejandra Maestripieri

Instituto Argentino de Matemática, CONICET.

Abstract

We study those orbits of oblique projections under the action of the full unitary group of a Hilbert space $\mathcal{H}$, which are submanifolds of $\mathcal{B}(\mathcal{H})$. We also consider orbits under the Schatten unitaries, and obtain a partial characterization of the submanifold condition for these orbits. Finsler metrics are introduced, and the minimality of metric geodesics is investigated.

1 Introduction

Given a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{G}(\mathcal{H})$ the group of invertible operators and $\mathcal{U}(\mathcal{H})$ the subgroup of unitary operators. An oblique projection is an idempotent $Q \in \mathcal{B}(\mathcal{H})$. This paper is devoted to a geometric study of the unitary orbit of a fixed oblique projection Q:

$$\mathcal{U}_Q = \{UQU^* : U \in \mathcal{U}(\mathcal{H})\}.$$

Of course, $\mathcal{U}_Q$ consists of oblique projections and it is a proper part of the similarity orbit

$$\mathcal{S}_Q = \{GQG^{-1} : G \in \mathcal{G}(\mathcal{H})\}.$$

Observe that $\mathcal{U}_Q$ is a bounded subset of $\mathcal{S}_Q$: the norm of every Q' in $\mathcal{U}_Q$ is $\|Q\|$. On the other side, except in the trivial cases where $Q = 0$ or 1, the similarity orbit is unbounded. If the literature on similarity orbits of oblique projections is very rich, the unitary orbit $\mathcal{U}_Q$ has been less studied, perhaps because of the "odd nature" of $\mathcal{U}_Q$: in some sense, one expects that the unitary group acts on self-adjoint elements, and oblique projections are rarely self-adjoint. Of course, if $Q^* = Q$, i.e., Q is an orthogonal projection, then the natural unitary orbit of Q has deserved much attention. It should be observed that unitary equivalence is in general a difficult problem, even for matrices. However, Jacques Dixmier proved that the study of $\mathcal{U}_Q$ is related to the problem of unitary equivalence of pair of closed subspaces, at least if the pair is in generic position ("position p", in French). Raeburn and Sinclair could simplify Dixmier's proof and found a way for avoiding the extra hypothesis on generic positions. We shall enlarge this comment at Section 2, where, after describing the unitary orbit of Q, we consider the problem of characterizing the oblique projections Q such that $\mathcal{U}_Q$ is a submanifold of $\mathcal{B}(\mathcal{H})$. It turns out that this occurs if and only if the spectrum of Q^*Q is finite. This is done at Section 3, using some fine results by G. K. Pedersen. At Section 4, we define a reductive structure and at Section 5 we define a Finsler metric on $\mathcal{U}_Q$. In the main result of this section we prove the minimality of a large family of geodesics, where minimal means that, in the length notion defined by the Finsler metric, geodesics are short. The two last sections contain a description of the action of Schatten unitary groups and its orbits, and some minimality results in these smaller orbits.

*2000 MSC. Primary 22E65; Secondary 58E50, 58B20.

2 The unitary orbit of Q

Given an oblique projection $Q \in \mathcal{B}(\mathcal{H})$, the *unitary orbit* of Q is the set of projections which are unitary similar to Q, i.e. $\mathcal{U}_Q = \{UQU^* : U \in \mathcal{U}(\mathcal{H})\}$. Observe that the problem of characterizing this set is equivalent to the problem of finding the unitary invariants of the pair of subspaces $\{R(Q), N(Q)\}$, where $R(Q)$ and $N(Q)$ are the range and the nullspace of Q, respectively; or, equivalently, to find the pairs of orthogonal projections $\{P_1, P_2\}$ such that there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $P_1 = UP_{R(Q)}U^*$ and $P_2 = UP_{N(Q)}U^*$. Two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ are in *generic position* if

$$\mathcal{M} \cap \mathcal{N} = \mathcal{M} \cap \mathcal{N}^\perp = \mathcal{M}^\perp \cap \mathcal{N} = \mathcal{M}^\perp \cap \mathcal{N}^\perp = \{0\}.$$

In [8], J. Dixmier proved that if the closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ are in generic position, then the pair of subspaces $\{\mathcal{M}', \mathcal{N}'\}$ is unitarily equivalent to the pair $\{\mathcal{M}, \mathcal{N}\}$ (i.e. there exists $U \in \mathcal{U}(\mathcal{H})$ such that $U(\mathcal{M}) = \mathcal{M}'$ and $U(\mathcal{N}) = \mathcal{N}'$ if and only if the operators $P_\mathcal{M} + P_\mathcal{N}$ and $P_{\mathcal{M}'} + P_{\mathcal{N}'}$ are unitarily equivalent. See a different proof by P.R. Hamos [13].

Later, I. Raeburn and A. M. Sinclair [21] generalized this result to subspaces which are not necessarily in generic position. More precisely, they proved that if $\{P, Q\}$ and $\{P', Q'\}$ are two pairs of orthogonal projections and $\lambda > 1$, then there is a unitary operator U such that $UPU^* = P'$ and $UQU^* = Q'$ if and only if the positive operator $\lambda P + Q$ is unitarily equivalent to $\lambda P' + Q'$.

We present here an alternative description of the unitary orbit of an oblique projection $Q_0 \in \mathcal{B}(\mathcal{H})$. Denote $P_0 = P_{R(Q_0)}$. In matrix form

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_0 = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}.$$

Note that two oblique projections Q_0 and Q_1,

$$Q_0 = \begin{pmatrix} 1 & B_0 \\ 0 & 0 \end{pmatrix} \begin{matrix} R(Q_0) \\ R(Q_0)^\perp \end{matrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} 1 & B_1 \\ 0 & 0 \end{pmatrix} \begin{matrix} R(Q_1) \\ R(Q_1)^\perp \end{matrix}$$

are unitarily equivalent if and only if there exist surjective isometries $U_{01} : R(Q_0) \to R(Q_1)$ and $U_{10} : R(Q_0)^\perp \to R(Q_1)^\perp$ such that $U_{01}B_0 = B_1U_{10}$. Indeed, if $Q_1 = UQ_0U^*$, then also $P_1 = UP_0U^*$. In particular, this implies that U maps $R(Q_0)$ onto $R(Q_1)$, and the same for the orthogonal complements, i.e.,

$$U_{01} = P_1UP_0 : R(Q_0) \to R(Q_1) \text{ and } U_{10} = (1 - P_1)U(1 - P_0) : R(Q_0)^\perp \to R(Q_1)^\perp$$

are isometries. On the other hand, also the matrices

$$\begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} \begin{matrix} R(Q_0) \\ R(Q_0)^\perp \end{matrix} \quad \text{and} \quad \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} \begin{matrix} R(Q_1) \\ R(Q_1)^\perp \end{matrix}$$

are unitarily equivalent by means of the same U. As straightforward matrix computation shows that $U_{01}B_0U_{10} = B_1$.

Conversely, if such isometries U_{01} and U_{01} exist, it implies that P_0 and P_1 are unitarily equivalent by means of $U = U_{01} \oplus U_{10}$, which therefore also implements the equivalence between Q_0 and Q_1.

For canonical forms of projections in finite dimensional spaces, which are unitarily equivalent to a fixed oblique projection, the reader is referred to the papers by Dokovic [10] and Ikramov [15].

3 A smooth structure for the unitary orbit of Q

We shall characterize first the oblique projections Q such that their unitary orbits are complemented submanifolds of $\mathcal{B}(\mathcal{H})$, in terms of the operators B. We fix some notation.

Let $\mathcal{U}_{Q_0} = \{UQ_0U^* : U \in \mathcal{U}(\mathcal{H})\}$. Denote by π_{Q_0} the map

$$\pi_{Q_0} : \mathcal{U}(\mathcal{H}) \to \mathcal{U}_{Q_0} \quad \pi_{Q_0}(U) = UQU^*.$$

Theorem 3.1. *The orbit $\mathcal{U}_{Q_0} \subset \mathcal{B}(\mathcal{H})$ is a real analytic submanifold if and only if the spectrum of $Q_0^*Q_0$ is finite. In that case, the map π_{Q_0} is a real analytic submersion.*

Proof. First note that the condition that the spectrum of $Q_0^*Q_0$ is finite is equivalent to the condition that the spectrum of B^*B (or of BB^*) is finite. Indeed

$$Q_0Q_0^* = \begin{pmatrix} 1+BB^* & 0 \\ 0 & 0 \end{pmatrix},$$

and thus the spectrum of $Q_0Q_0^*$ (or of $Q_0^*Q_0$) is finite if and only if the spectrum of BB^* (or B^*B) is finite.

In [4] it was shown that the unitary orbit of an operator is a submanifold of $\mathcal{B}(\mathcal{H})$ if and only if the C^*-algebra $\mathcal{C}^*(Q_0)$ generated by 1 and Q is finite dimensional. Also it was shown that in this case the map π_{Q_0} is a submersion.

Note that $P_0 \in \mathcal{C}^*(Q_0)$; indeed, it is a known fact that P_0 can be obtained, for instance, as $P_0 = Q_0Q_0^*(1 + (Q_0 - Q_0^*)^2)^{-1}$. Then

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} = Q_0(1 - P_0) + (1 - P_0)Q_0^* = D \in \mathcal{C}^*(Q_0).$$

Thus

$$\begin{pmatrix} BB^* & 0 \\ 0 & B^*B \end{pmatrix} = D^2 \in \mathcal{C}^*(Q_0).$$

If $\mathcal{C}^*(Q_0)$ is finite dimensional, any self-adjoint element in $\mathcal{C}^*(Q_0)$ must have finite spectrum (otherwise, by a simple spectral calculus argument, one can find infinitely many linearly independent elements in $\mathcal{C}^*(Q_0)$). Thus, since $\sigma(D^2) = \sigma(BB^*) \cup \sigma(B^*B)$ (and $\sigma(BB^*) \cup \{0\} = \sigma(B^*B) \cup \{0\}$), it follows that $\sigma(B^*B)$ is finite.

In [16], it was shown that the algebra generated by an idempotent, is generated by two self-adjoint projections, in our case

$$P_0\ , \quad \text{and } R = \begin{pmatrix} 1 - CC^* & (1 - CC^*)^{1/2}C \\ C^*(1 - CC^*)^{1/2} & C^*C \end{pmatrix},$$

where $C = \frac{1}{2\|B\|}B$. In [18], G.K. Pedersen showed that the C^*-algebra $\mathcal{C}^*(Q_0)$, generated by two projections P_0 and R is determined by $\sigma = \sigma(P_0RP_0) = \sigma(1 - CC^*)$, which is finite in our case. Let $\sigma' = \sigma - \{0\}$ and $\sigma'' = \sigma - \{1\}$. In Theorems 3.2 and 3.4 of [18], Pedersen proved that if

$$\mathcal{A} = \begin{pmatrix} \mathcal{C}_0(\sigma') & \mathcal{C}_0(\sigma'') \\ \mathcal{C}_0(\sigma'') & \mathcal{C}_0(\sigma'') \end{pmatrix},$$

then:

1. If $0 \notin \bar{\sigma'}$, then $\mathcal{C}^*(Q)$ is isomorphic to $\mathcal{A}$, $\mathcal{A} \oplus \mathbb{C}$ or $\mathcal{A} \oplus \mathbb{C} \oplus \mathbb{C}$.

2. If $0 \in \bar{\sigma'}$, then

$$\mathcal{C}^*(Q) \simeq \begin{pmatrix} \mathcal{C}(\sigma) & \mathcal{C}_0(\sigma'') \\ \mathcal{C}_0(\sigma'') & \mathcal{C}_0(\sigma - \{1\}) \end{pmatrix}.$$

Since σ is a finite set, all these algebras are finite dimensional. □

Remark 3.2. Note that if $\mathcal{C}^*(Q)$ is finite dimensional, then B is of the form $B = \sum_{i=1}^n \lambda_i V P_i$, where P_i are mutually orthogonal projections and V is a partial isometry with initial space $\sum_{i=1}^n P_i$. For this class of oblique projections, unitary equivalence is determined by the numbers $\lambda_1, \ldots, \lambda_n$ and their multiplicities (i.e., the ranks of the projections P_i)

Examples of oblique projections such that $\mathcal{U}_{Q_0}$ is a submanifold are obtained if, for instance B_0 is a partial isometry. If one fixes the range P_0, an oblique projection Q (with a partial isometry in the 12 entry (in terms of the decomposition $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_0$) lies in the unitary orbit of Q_0 if and only if the partial isometries B_0 and B are unitarily equivalent in the sense of [1]: there exist unitaries U, W acting in $\mathcal{H}_0$ such that $B = UB_0W^*$.

In any case, $\mathcal{U}_{Q_0}$ can be endowed with the quotient topology and a quotient smoooth structure which makes π_{Q_0} a submersion. Following the notation in [5],

Definition 3.3. *A subgroup $G \subset \mathcal{U}(\mathcal{H})$ is called an algebraic subgroup of degree $\leq n$ if there exists a family of polynomials $\mathcal{P}$ in $\mathcal{B}(\mathcal{H})$ in two variables of degree $\leq n$ such that*

$$G = \{U \in \mathcal{U}(\mathcal{H}) : p(U, U^*) = 0 \quad \textit{for all } p \in \mathcal{P}\}.$$

A polynomial p in $\mathcal{B}(\mathcal{H})$ in two variables is a sum $p(A, B) = \sum_{i=1}^k \psi_k(A, B)$, $A, B \in \mathcal{B}(\mathcal{H})$ where each ψ_i is given by a monomial of degree $d_i \leq n$. A monomial ψ of degree d is given by a d-multilinear map

$$\rho : (\mathcal{B}(\mathcal{H})^2)^d \to \mathcal{B}(\mathcal{H}),$$

by means of $\psi(A, B) = \rho((A, B), \ldots, (A, B))$.

In the case of $\mathcal{U}_{Q_0}$, the isotropy subgroup of Q_0 is $G_{Q_0} = \{U \in \mathcal{B}(\mathcal{H}) : UQ_0U^* = Q_0\}$. Consider the polynomial $p(A, B) = AQ_0B - Q_0$. Then clearly G_{Q_0} is an algebraic subgroup of degree ≤ 1. Thus one can apply theorems 4.18 and 4.19 in [5] to obtain

Proposition 3.4. *The unitary orbit $\mathcal{U}_{Q_0}$, endowed with the quotient topology $\mathcal{U}(\mathcal{H})/G_{Q_0}$, has a structure of real analytic manifold, and the map π_{Q_0} is a real analytic submersion.*

In any case, the tangent spaces of $\mathcal{U}_Q$ at Q_1

$$(T\mathcal{U}_Q)_{Q_1} = \{XQ_1 - Q_1X : X \in \mathcal{B}(\mathcal{H})_{ah}\}. \tag{1}$$

If $\mathcal{U}_Q$ is a submanifold, the space is considered with the norm topology. Otherwise, it should be considered with the quotient norm topology: $\|XQ_1 - Q_1X\| = \inf\{\|Y\| \in Y \in \mathcal{B}(\mathcal{H})_{ah} : YQ_1 - Q_1Y = XQ_1 - Q_1Y\}$

4 Reductive structure

The isotropy group G_Q of the action is the unitary group of the commutant of $\mathcal{C}^*(Q)$. Suppose that $\mathcal{C}^*(Q)$ is finite dimensional. Then there exist Hilbert spaces $\mathcal{J}_1, \ldots, \mathcal{J}_k$ and positive integers $n_1, \ldots, n_k$ such that

$$\mathcal{H} \simeq \mathcal{J}_1^{n_1} \oplus \ldots \oplus \mathcal{J}_k^{n_k}, \tag{2}$$

and

$$\mathcal{C}^*(Q) \simeq (M_{n_1}(\mathbb{C}) \otimes Id_{\mathcal{J}_1}) \oplus \ldots \oplus (M_{n_k}(\mathbb{C}) \otimes Id_{\mathcal{J}_k}).$$

That is, in terms of the decomposition 2 above, an element $A \in \mathcal{C}^*(Q)$ has a matrix of the form

$$A = \begin{pmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ 0 & 0 & A_3 & \ldots & 0 \\ \ldots & & & & \\ 0 & \ldots & & 0 & A_k \end{pmatrix} \begin{matrix} \mathcal{J}_1^{n_1} \\ \mathcal{J}_2^{n_2} \\ \mathcal{J}_3^{n_3} \\ \\ \mathcal{J}_k^{n_k} \end{matrix} ,$$

where each entry a_l is a scalar matrix of the form $A_l = (a^l_{i,j})_{1\le i,j\le n_l} \cdot I_{\mathcal{J}_l}$.

Therefore the isotropy algebra $\mathcal{G}_Q$, which is the skew-adjoint part of the commutant of $\mathcal{C}^*(Q)$, consists of matrices of the form

$$B = \begin{pmatrix} B_1 & 0 & 0 & \ldots & 0 \\ 0 & B_2 & 0 & \ldots & 0 \\ 0 & 0 & B_3 & \ldots & 0 \\ \ldots & & & & \\ 0 & \ldots & & 0 & B_k \end{pmatrix} \begin{matrix} \mathcal{J}_1^{n_1} \\ \mathcal{J}_2^{n_2} \\ \mathcal{J}_3^{n_3} \\ \\ \mathcal{J}_k^{n_k} \end{matrix} ,$$

where each B_l is a skew-adjoint operator in $\mathcal{J}_l^{n_l}$, whose $n_l \times n_l$ matrix is of the form

$$B_l = \begin{pmatrix} B'_l & 0 & 0 & \ldots & 0 \\ 0 & B_l & 0 & \ldots & 0 \\ 0 & 0 & B'_l & \ldots & 0 \\ \ldots & & & & \\ 0 & \ldots & & 0 & B'_l \end{pmatrix} .$$

A reductive structure consists of a supplement $\mathcal{K}_Q$ for the isotropy algebra $\mathcal{G}_Q$ which is inner invariant for the action of G_Q: $V\mathcal{K}_Q V^* = \mathcal{K}_Q$. In view of the matrix form of $\mathcal{G}_Q$, clearly if amounts to finding a supplement for the skew-adjoint operators b_l above, in each diagonal block (acting in $\mathcal{J}_l^{n_l}$). Consider the space $\mathcal{K}_l$ of skew-adjoint operators in $\mathcal{J}_l^{n_l}$ given by

$$\mathcal{K}_l = \{Y = (Y_{i,j})_{1\le i,j\le n_l} \in \mathcal{B}(\mathcal{J}_l^{n_l}) : Y^* = -Y \text{ and } \sum_{i=1}^{n_l} Y_{ii} = 0\}. \quad (3)$$

Denote $\mathcal{H}_l = \mathcal{J}_l^{n_l}$. Thus $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k$. Then the supplement $\mathcal{K}_Q$ is

$$\mathcal{K}_Q = \{Z = (Z_{i,j})_{1\le i,j\le k} \in \mathcal{B}(\mathcal{H}) : Z^* = -Z \text{ and } Z_{l,l} \in \mathcal{K}_l \ , \ 1 \le l \le k\}.$$

Straightforward matrix computations show that

1. $$\mathcal{B}(\mathcal{H})_{ah} = \mathcal{G}_Q \oplus \mathcal{K}_Q.$$

2. $$V\mathcal{K}_Q V^* = \mathcal{K}_Q \ \text{ for any } \ v \in G_Q. \quad (4)$$

. If $Q_1 = UQU* \in \mathcal{U}_Q$, put $\mathcal{K}_{Q_1} = U\mathcal{K}_Q U^*$. It is apparent that $\mathcal{K}_{Q_1}$ is a supplement for $\mathcal{G}_{Q_1}$. The definition of $\mathcal{K}_{Q_1}$ does not depend on the choice of U due to (4).

A reductive structure defines a linear connection in $\mathcal{U}_Q$; this is a standard fact in the theory of homogeneous spaces (see [14] as a classical reference, or [17] where the specific case of homogeneous spaces of operators is treated). For instance, if $Q_1 \in \mathcal{U}_Q$ and $XQ - QX \in (T\mathcal{U}_Q)_{Q_1}$, then there is a unique $Z \in \mathcal{K}_{Q_1}$ such that $ZQ_1 - Q_1Z = XQ_1 - Q_1X$. Then the unique geodesic δ of the connection with $\delta(0) = Q_1$ and $\dot{\delta}(0) = XQ_1 - Q_1X$ is given by

$$\delta(t) = e^{tZ}Q_1e^{-tZ}.$$

5 A Finsler metric for $\mathcal{U}_Q$

Following ideas and methods in [11], we shall consider the following norms in the tangent spaces of $\mathcal{U}_Q$. If $Q_1 \in \mathcal{U}_Q$ and $v = XQ_1 - Q_1X \in (T\mathcal{U}_Q)_{Q_1}$, put

$$\|v\|_{Q_1} = \inf\{\|Y\| : Y^* = -Y \text{ and } YQ - QY = v\}. \tag{5}$$

Each $Y \in \mathcal{B}(\mathcal{H})_{ah}$ such that $v = YQ - QY$ is called a *lifting* of v. A lifting Z of v is called *minimal* if it achieves the infimum above, i.e. $\|Z\| = \|z\|_{Q_1}$. A simple argument shows that minimal liftings always exist. Indeed, it is obtained as a weak limit point of a minimizing sequence, as follows. Let $Y_n^* = -Y_n$ with $Y_nQ_1 - Q_1Y_n = v$ such that $\|Y_n\| \to \|v\|_{Q_1}$. The sequence $\{Y_n\}$ is bounded, therefore thre exists subsequence, still denoted Y_n, which is convergent in the weak operator topology, $< Y_n\xi, \eta > \to < Z\xi, \eta >$ for all $\xi, \eta \in \mathcal{H}$. Note that

$$v = Y_nQ_1 - Q_1Y_n \overset{wot}{\to} ZQ_1 - Q_1Z,$$

and thus $ZQ_1 - Q_1Z = v$. Also it is clear that $Z^* = -Z$. For $\epsilon > 0$ fix a unit vector ξ such that $| < Z\xi, \xi > | \geq \|Z\| - \epsilon$. Then

$$\|Y_n\| \geq | < Y_n\xi, \xi > | \to | < Z\xi, \xi > | \geq \|Z\| - \epsilon,$$

and therefore taking limits $\|Z\| \geq \|v\|_{Q_1} \geq \|Z\| - \epsilon$. We remark that the minimal lifting may not be unique.

Minimal liftings are relevant due to the following result by Mata, Durán and Recht [11]

Theorem 5.1. *Let $Q_1 \in \mathcal{U}_Q$ and $v \in (T\mathcal{U}_Q)_{Q_1}$, with $Z \in \mathcal{B}(\mathcal{H})_{ah}$ a minimal lifting of v. Then the curve $\delta(t) = e^{tZ}Q_1e^{-tZ}$ has minimal length for $|t| \leq \frac{\pi}{2\|Z\|} = \frac{\pi}{2\|v\|_{Q_1}}$.*

In general, minimal liftings are not easy to compute (see for instance [3] as an example of the complexity of this problem even in finite dimensions). Note also that the *metric geodesic* (or curve that has minimal length) described above looks formally similar to the geodesic of the linear connection in $\mathcal{U}_Q$ described in the previous section. However the exponents Z in both curves are different. It is seldom the case that the geodesics of the linear connection are minimal with respect to the Finsler metric given by the operator norm (see [3]). One remarkable exception to this observation is the homogeneous space of *self-adjoint* projections of a C*-algebra ([6], [19]): in this case minimal liftings are unique and coincide with the elements in the reductive supplement of the isotropy Lie algebra.

Let us consider a simple case of an oblique projection, where more can be said on the minimal liftings of certain special tangent directions. Let $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_0$ and Q be

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_0 \end{matrix} .$$

Note that the unitary orbit of this Q consists of all oblique projections

$$Q_1 = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix} \begin{matrix} R(Q_1) \\ Q(Q_1)^\perp \end{matrix}$$

with $B : R(Q_1)^\perp \to R(Q_1)$ isometric.

The isotropy algebra $\mathcal{G}_Q$ and the reductive supplement $\mathcal{K}_Q$ consist of matrices which are of the form, respectively,

$$X = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix},$$

with A, D self-adjoint in $\mathcal{H}_0$. We shall say that a tangent vector $v \in (T\mathcal{U}_Q)_Q$ is *symmetric* if, in the corresponding element $Z \in \mathcal{K}_Q$ (characterized by $ZQ - QZ = v$), B is skew-adjoint. This condition means that v has a lifting Z of the form

$$Z = \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

with all entries skew-adjoint. A vector v tangent at $Q_1 = UQU^*$ is called symmetric, if U^*vU (which is tangent at Q) is symmetric in the above sense. Note that this does not depend on the choice of u.

Let us show that symmetric tangent vectors have reductive symbols Z which are minimal liftings. It suffices to show this fact at Q.

Lemma 5.2. *Let $v = ZQ - QZ$ be symmetric, $Z \in \mathcal{K}_Q$ with $B^* = -B$. Then Z is a minimal lifting of v. Namely, $\|Z\| \leq \|Z + X\|$, for all $X \in \mathcal{G}_Q$.*

Proof. Multiplying all operators by i, enables one to reason with self-adjoint operators. Suppose then that all operators concerned are self-adjoint (and we use the same letters to name them). Let $\xi = (\xi_1, \xi_2) \in \mathcal{H}_0 \times \mathcal{H}_0$ with $\|\xi\| = 1$. Then

$$< Z\xi, \xi >=< A\xi_1, \xi_1 > - < A\xi_2, \xi_2 > +2Re < B\xi_2, \xi_1 > .$$

Note that if $\eta = (-\xi_2, \xi_1)$, then $\|\eta\| = 1$, and

$$< A\eta, \eta >= - < A\xi, \xi > .$$

The key fact here is that $B^* = B$. It follows that both $-\|Z\|$ and $+\|Z\|$ belong to the spectrum of Z. Let $\xi^n = (\xi_1^n, \xi_2^n) \in \mathcal{H}_0 \times \mathcal{H}_0$ with $\|\xi^n\| = 1$, such that $< Z\xi^n, \xi^n >\to \|Z\|$. Taking $\eta^n = (-\xi_2^n, \xi_1^n)$ as above, one has that $< Z\eta^n, \eta^n >\to -\|Z\|$. If $X \in \mathcal{G}_Q$, by the remarks above,

$$< (Z + X)\xi^n, \xi^n >=< Z\xi^n, \xi^n > + < D\xi_1^n, \xi_1^n > + < D\xi_2^n, \xi_2^n > .$$

Also note that $< X\eta^n, \eta^n >=< X\xi^n, \xi^n >= r_n$, which is a bounded sequence in $\mathbb{R}$. Consider a convergent subsequence of these numbers, and denote it again by r_n, with $r_n \to r_0$. Then

$$\|Z + X\| \geq< (Z + X)\xi^n, \xi^n >\to \|Z\| + r_0,$$

and

$$-\|Z + X\| \leq< (Z + X)\eta^n, \eta^n >\to -\|Z\| + r_0,$$

Therefore if either $r_0 \geq 0$ or $r_0 < 0$, one has that $\|Z + X\| \geq \|Z\|$. ☐

Let us denote by $\mathbb{S}T(\mathcal{U}_Q)_{Q_1}$ be the space of symmetric tangent vectors. Apparently, it is a closed complemented subspace of $T(\mathcal{U}_Q)_{Q_1}$. Indeed, note that if we denote by δ_{Q_1} the differential of the map $\pi_{Q_1} : \mathcal{U}(\mathcal{H}) \to \mathcal{U}_Q$, $\pi_{Q_1}(U) = UQ_1U^*$ at the identity 1,

$$\delta_{Q_1} : \mathcal{B}(\mathcal{H})_{ah} \to (T\mathcal{U}_q)_{Q_1}, \quad \delta_{Q_1}(X) = XQ_1 - Q_1X,$$

then $N(\delta_{Q_1}) = \mathcal{G}_{Q_1}$ and therefore

$$\delta_{Q_1}|_{\mathcal{K}_{Q_1}} : \mathcal{K}_{Q_1} \to (T\mathcal{U}_Q)_{Q_1}$$

is an isomorphism (its inverse is usually called the 1-form of the linear connections [14], [17]). Note that $\mathbb{S}T(\mathcal{U}_Q)_{Q_1}$ is the image of the subspace $\{Z \in \mathcal{K}_{Q_1} : B^* = -B\} \subset \mathcal{K}_{Q_1}$ under this isomorphism. An arbitrary $Z \in \mathcal{K}_{Q_1}$ can be decomposed as

$$Z = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix} = \begin{pmatrix} A & B_{ah} \\ B_{ah} & -A \end{pmatrix} + \begin{pmatrix} 0 & B_h \\ -B_h & 0 \end{pmatrix} = Z_{\mathbb{S}} + Z_{\mathbb{D}},$$

where B_{ah} and B_h are skew-adjoint and self-adjoint parts of B, respectively. The image of all matrices of the same form as the second summand under δ_{Q_1} is a supplement for $\mathbb{S}(T\mathcal{U}_q)_{Q_1}$, which we shall call $\mathbb{D}(T\mathcal{U}_q)_{Q_1}$. Let us show that the liftings of vectors in $\mathbb{D}(T\mathcal{U}_q)_{Q_1}$ are also minimal.

Lemma 5.3. *Let $v \in \mathbb{D}(T\mathcal{U}_q)_{Q_1}$, $v = ZQ_1 - Q_1Z$ with $Z = Z_{\mathbb{D}}$ as above. Then for all $X \in \mathcal{G}_{Q_1}$,*

$$\|Z\| \leq \|Z + X\|.$$

Proof. The proof is similar as in the previous case, even more simple. Again suppose the operators Z and X are self-adjoint. If $\xi = (\xi_1, \xi_2) \in \mathcal{H}$, then

$$< Z\xi, \xi >= 2Re < B\xi_2, \xi_1 >,$$

and clearly the set of these values is symmetric with respect to the origin (for instance, if $\xi' = (-\xi_1, \xi_2)$, $< Z\xi', \xi' >= - < Z\xi, \xi >$), and the proof follows as in the previous case. □

We may summarize these facts in the following:

Theorem 5.4. *Let $Q_1 \in \mathcal{U}_q$ and $v \in (T\mathcal{U}_q)_{Q_1}$. Then if either v lies in $\mathbb{S}(T\mathcal{U}_q)_{Q_1}$ or its supplement $\mathbb{D}(T\mathcal{U}_q)_{Q_1}$, then the unique geodesic of the connection with $\delta(0) = Q_1$ and $\dot{\delta}(0) = v$, given by $\delta(t) = e^{tZ}Q_1e^{-tZ}$, for $Z \in \mathcal{K}_{Q_1}$ ($ZQ_1 - Q_1Z = V$), is minimal for all t such that $|t| \leq \frac{\pi}{2\|v\|_{Q_1}} = \frac{\pi}{2\|Z\|}$.*

6 The action of the Schatten unitary groups

In this section we consider the former action restricted to the p-Schatten unitary group $\mathcal{U}_p(\mathcal{H})$, for a fixed $1 \leq p \leq \infty$

$$\mathcal{U}_p(\mathcal{H}) = \{U \in \mathcal{U}(\mathcal{H}) : U - 1 \in \mathcal{B}_p(\mathcal{H})\},$$

where $\mathcal{B}_p(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : Tr(|X|^p) < \infty\}$ and $\mathcal{B}_\infty(\mathcal{H})$ denote the ideals of p-Schatten and compact operators, respectively. Denote by $\mathcal{U}_{p,Q}$ the restricted orbit of Q,

$$\mathcal{U}_{p,Q} = \{UQU^* : U \in \mathcal{U}_p(\mathcal{H})\}.$$

Note that $\mathcal{U}_{p,Q} = \{UQU^* : U \in \mathcal{U}_p(\mathcal{H})\} \subset Q + \mathcal{B}_p(\mathcal{H}) = \{Q + X : X \in \mathcal{B}_p(\mathcal{H})\}$. Indeed,

$$UQU^* = Q + (U-1)QU^* + UQ(U^* - 1) + (U-1)Q(U^* - 1) \in Q + \mathcal{B}_p(\mathcal{H}).$$

This space $Q+\mathcal{B}_p(\mathcal{H})$ can be endowed with the p-metric: $d(Q+X, Q+Y) = \|X-Y\|_p$. It can be regarded as an affine Banach space. Adapting techniques from [4], it can be proved that if $\mathcal{C}^*(Q)$ is finite dimensional, then the restricted orbit $\mathcal{U}_{p,Q}$ is a submanifold of $Q + \mathcal{B}_p(\mathcal{H})$.

First we need the following Lemmas. The first was stated in the appendix of [20].

Lemma 6.1. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x \in X$, denote by $\pi_x : G \to X$ the smooth map $\pi_x(g) = g \cdot x$. Suppose that*

1. *π_x is an open mapping, when regarded as a map from G onto the orbit $\{g \cdot x : g \in G\}$ of x (with the relative topology of X).*
2. *The differential $d(\pi_x)_1 : (TG)_1 \to X$ splits: its kernel and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x : g \in G\}$ is a smooth submanifold of X, and the map $\pi_x : G \to \{g \cdot x : g \in G\}$ is a smooth submersion.

Lemma 6.2. *Suppose that there exists a real analytic map*

$$\bar{\Omega} : \mathcal{B} \to 1 + \mathcal{B}_p(\mathcal{H})$$

defined on an open neighbourhood $\mathcal{B}$ of Q in $Q + \mathcal{B}_p(\mathcal{H})$, such that the restriction $\Omega = \bar{\Omega}|_{\mathcal{U}_{p,Q}}$ takes values in $\mathcal{U}_p(\mathcal{H})$ and is a cross section for $\pi_Q : \mathcal{U}_p(\mathcal{H}) \to \mathcal{U}_{p,Q}$, with $\bar{\Omega}(Q) = 1$. Then $\mathcal{U}_{p,Q} \subset Q + \mathcal{B}_p(\mathcal{H})$ is a real analytic submanifold and π_Q is a real analytic submersion.

Proof. Clearly, $\pi_Q : \mathcal{U}_p(\mathcal{H}) \to \mathcal{U}_{p,Q}$ is open, having local cross sections. Also it is real analytic, regarded as a map from with values in $\mathcal{B}_p(\mathcal{H})$. As before, denote $(d\bar{\pi}_Q)_1 = \delta_Q$. Since $\bar{\Omega}(Q) = 1$, we may eventually shrink $\mathcal{B}$ in order that $\bar{\Omega}$ takes invertible values (in the unitization on $\mathcal{B}_p(\mathcal{H})$). The polar decomposition of invertible operators in the unitization of $\mathcal{B}_p(\mathcal{H})$ is a real analytic operation; more precisely, both the unitary part and the absolute value are real analytic maps. By composing $\bar{\Omega}$ with the unitary part, we may suppose that furthermore $\bar{\Omega}$ takes unitary values. Note two facts. First, since by hypothesis $\bar{\Omega}$ takes values in $1 + \mathcal{B}_p(\mathcal{H})$, the unitary values (equal to the unitary part of $\bar{\Omega}$) belong to $\mathcal{U}_p(\mathcal{H})$. Second, taking unitary part has no effect when one restricts $\bar{\Omega}$ to $\mathcal{U}_{p,Q}$.

We claim the following facts:

1.
$$(d\bar{\pi}_Q \circ \bar{\Omega} \circ \bar{\pi}_Q)_1(X) = \delta_Q \circ (d\bar{\Omega})_Q \circ \delta_Q(X) = \delta_Q(X)$$
and

2.
$$(d\bar{\Omega})_Q \circ \delta_Q(X) \in \mathcal{B}_p(\mathcal{H})_{ah}.$$

The second fact is apparent, since $\bar{\Omega}$ takes values in $\mathcal{U}_p(\mathcal{H})$ and $\bar{\Omega}(Q) = 1$. To prove the first fact, consider $\gamma(t) = e^{tX} Q e^{-tX}$. Then

$$(d\bar{\pi}_Q \circ \bar{\Omega} \circ \bar{\pi}_Q)_1(X) = \frac{d}{dt}\bar{\pi}_Q \circ \bar{\Omega}(\gamma(t))|_{t=0} = \frac{d}{dt}\pi_Q \circ \Omega(\gamma(t))|_{t=0} = XQ - QX.$$

It follows that

$$\Delta := (d\bar{\Omega})_Q \circ \delta_Q : \mathcal{B}_p(\mathcal{H})_{ah} \to \mathcal{B}_p(\mathcal{H})_{ah},$$

verifies $\Delta^2 = \Delta$. Note that $N(\Delta) = N(\delta_Q)$, which implies that this subspace is complemented in $\mathcal{B}_p(\mathcal{H})_{ah}$.

On the other hand $R(\delta_Q \circ (d\bar{\Omega})_Q) \subset R\delta_Q$, because $(d\bar{\Omega})_Q)$ takes skew-adjoint values. Also it is clear that

$$\delta_Q \circ (d\bar{\Omega})_Q|_{R(\delta_Q)} = Id_{R(\delta_Q)}.$$

This implies that $\delta_Q \circ (d\bar{\Omega})_Q$ is an idempotent with range equal to $R(\delta_Q)$, and therefore also this space is complemented.

Then Lemma 6.1 applies. □

Proposition 6.3. *If* $\dim \mathcal{C}^*(Q) < \infty$, *then* $\mathcal{U}_{p,Q} \subset Q + \mathcal{B}_p(\mathcal{H})$ *is a real analytic submanifold, and the map* $\pi_Q : \mathcal{U}_p(\mathcal{H}) \to \mathcal{U}_{p,Q}$ *is a real analytic submersion.*

Proof. Let us construct local cross sections for π_Q, which are restrictions of real analytic maps defined on open sets in $Q + \mathcal{B}_p(\mathcal{H})$. The proof follows applying Lemma (6.2) above. We proceed as in [4], Th. 1.3. Since $\mathcal{C}^*(Q)$ is finite dimensional, there exist positive integers $n_1, \dots n_h$ and a $*$-isomorphism

$$\theta : \mathcal{C}^*(Q) \to M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_h}(\mathbb{C}).$$

Let $n = \sum_{i=1}^h n_i$. Consider the set of systems of projections

$$\mathcal{P}^n = \{(P_1, \dots P_n) \in \mathcal{B}(\mathcal{H})^n : P_i^2 = P_i^* = P_i,\ P_iP_j = 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^n P_i = 1\}.$$

Let $e^i_{j,k} \in M_{n_i}(\mathbb{C})$ be the elementary matrix with 1 in the j,k-entry and zero elsewhere (consider $M_{n_i}(\mathbb{C})$ imbedded in $M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_h}(\mathbb{C})$). Since $\theta(Q)$ and $\theta(Q)^*$ generate $M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_h}(\mathbb{C})$, it follows that there exist polynomials $p^i_{j,k}(\mathbb{X}, \mathbb{Y})$ in two non-commuting variables $\mathbb{X}, \mathbb{Y}$ such that $e^i_{j,k} = p^i_{j,k}(\theta(Q), \theta(Q)^*)$. Consider the operators $E^i_{J,k} \in \mathcal{B}(\mathcal{H})$ given by

$$E^i_{j,k} = \theta^{-1}(e^i_{j,k}) = p^i_{j,k}(Q, Q^*).$$

Consider the map

$$\varphi : \mathcal{U}_{p,Q} \to \mathcal{P}^n,$$

$$\varphi(X) = (p^1_{1,1}(X, X^*), p^1_{2,2}(X, X^*), \dots, p^1_{n_1,n_2}(X, X^*), \dots, p^h_{1,1}(X, X^*), \dots, p^h_{n_h,n_h}(X, X^*)).$$

Clearly it is a real analytic map, and it is well defined (it takes values in $\mathcal{P}_n$) because

$$p^i_{j,k}(UQU^*, UQ^*U^*) = Up^i_{j,k}(Q, Q^*)U^*,$$

and therefore $\varphi(UQU^*) = U\varphi(Q)U^*$. We consider the subset of restricted systems of projections,

$$\mathcal{P}^n_{p,Q} = \{(U\varphi_1(Q)U^*, \dots, U\varphi_n(Q)U^* : U \in \mathcal{U}_p(\mathcal{H})\}.$$

Apparently φ takes values in $\mathcal{P}^n_{p,Q}$. This space can be endowed with the p-metric in each coordinate. Indeed, each coordinate $\varphi_i(X)$ lies in $\varphi_i(Q) + \mathcal{B}_p(\mathcal{H})$. Let Σ be the following map defined on a neighbourhood of $\varphi(Q)$ in $\mathcal{P}^n_{p,Q}$ (with this p-metric):

$$\Sigma(R_1, \dots, R_n) = \sum_{i=1}^n \varphi_i(Q)R_i[1 - (\varphi_i(Q) - R_i)^2]^{-1/2}.$$

The map Σ is defined on the open subset of $\varphi(Q)$ in $\mathcal{P}^n_{p,Q}$ given by the condition that $1 - (\varphi_i(Q) - R_i)^2$ are invertible for $i = 1, \dots, n$. It can be shown that Σ takes unitary values (see [4]), and has the following property: $\varphi(X) = \Sigma(X)\varphi(Q)\Sigma(X)^*$.

Put $\kappa(X) = \sum_{i=1}^h \sum_{j=1}^{n_i} p^i_{j,1}(X, X^*)E^i_{1,j}$. It can be shown that [4]

$$\Omega(X) = \Sigma(\varphi(X))^*\ \kappa\,(\Sigma(\varphi(X))X\Sigma(\varphi(X))^*)$$

is a local cross section of the unitary action, defined on the open set (in the p-metric) of the elements $X \in \mathcal{U}_{p,Q}$ such that $\Sigma(\varphi(X))$ is defined.

Let us prove that it takes values in $\mathcal{U}_p(\mathcal{H})$. First note that if $(R_1, \ldots, R_n) \in \mathcal{P}^n_{p,Q}$, then $R_i = U\varphi_i(Q)U^*$ for some $U \in \mathcal{U}_p(\mathcal{H})$, $U = 1 + K$, $K \in \mathcal{B}_p(\mathcal{H})$. Then

$$1 - (\varphi_i(Q) - R_i)^2 = 1 - U\varphi_i(Q)U^* - \varphi_i(Q) + U\varphi_i(Q)U^*\varphi_i(Q) - \varphi_i(Q)U\varphi_i(Q)U^*.$$

Replacing U by $1 + K$, it is apparent that $1 - (\varphi_i(Q) - R_i)^2 \in 1 + \mathcal{B}_p(\mathcal{H})$. Therefore, if it is invertible, by a straightforward spectral argument, the element $[1 - (\varphi_i(Q) - R_i)^2]^{-1/2} = 1 + K'_i \in 1 + \mathcal{B}_p(\mathcal{H})$ (in fact, it lies in the group $Gl_p(\mathcal{H})$ of invertibles operators S such that $S - 1 \in \mathcal{B}_p(\mathcal{H})$, which is the group of invertibles of a $*$-Banach algebra, the unitization $\mathcal{B}_p(\mathcal{H})$). Therefore

$$\Sigma(R_1, \ldots, R_n) = \sum_{i=1}^{n} U\varphi_i(Q)U^*\varphi_i(Q)(1 + K'_i) = \sum_{i=1}^{n} U\varphi_i(Q)U^*\varphi_i(Q) + K'',$$

where $K'' = \sum_{i=1}^{n} U\varphi_i(Q)U^*\varphi_i(Q)K'_i \in \mathcal{B}_p(\mathcal{H})$. Note that

$$\sum_{i=1}^{n} U\varphi_i(Q)U^*\varphi_i(Q) = 1 + \sum_{i=1}^{n}(K\varphi_i(Q)K^*\varphi_i(Q) + \varphi_i(Q)K^*\varphi_i(Q) + K\varphi_i(Q)) \in 1 + \mathcal{B}_p(\mathcal{H}).$$

Finally, if $X = UQU^*$ with $U = 1 + K \in \mathcal{U}_p(\mathcal{H})$, then

$$\kappa(X) = \sum_{i=1}^{h}\sum_{j=1}^{n_i} UE^i_{j,1}U^*E^i_{1,j} = K' + \sum_{i=1}^{h}\sum_{j=1}^{n_i} E^i_{j,1}E^i_{1,j} = K' + 1,$$

where $K' = \sum_{i=1}^{h}\sum_{j=1}^{n_i}(KE^i_{j,1}K^*E^i_{1,j} + E^i_{j,1}K^*E^i_{1,j}) + K$. It follows that $\kappa(X) \in \mathcal{U}_p(\mathcal{H})$, and thus Ω takes values in $\mathcal{U}_p(\mathcal{H})$.

In order to apply Lemma 6.2, we must show that Ω can be extended to a real analytic map defined on an open neighbourhood of Q in $Q + \mathcal{B}_p(\mathcal{H})$. This is apparent from the algebraic expression of Ω.

$\square$

Note that the Proposition above holds for any operator $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{C}^*(A)$ is finite dimensional.

There is a partial converse to this Proposition, that holds in the case where the (full) unitary orbit of Q contains an element

$$Q_0 = \begin{pmatrix} 1 & B_0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_0 \end{matrix}$$

with $B_0^* = B_0$. This condition is equivalent to $\dim N(B) = \dim N(B^*)$. Indeed, if $\dim N(B) = \dim N(B^*)$, then one can choose a unitary operator in the polar decomposition $B = U|B|$. Then

$$Q_0 = \begin{pmatrix} 1 & |B| \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix}.$$

Conversely, if there is one such Q_0 in the orbit of Q, by the discusion in section 2, there exist surjective isometries $U_{01} : R(Q_0) \to R(Q)$ and $U_{10} : R(Q_0)^\perp \to R(Q)^\perp$ such that $U_{01}B_0 = BU_{10}$, and therefore

$$\dim N(B) = \dim N(B_0) = \dim N(B_0^*) = \dim N(B^*).$$

Theorem 6.4. *Suppose that* $p \geq 2$*, and that* Q *is unitary equivalent to an oblique projection* Q_0 *with* B_0 *self-adjoint. Then* $\mathcal{U}_{p,Q} \subset Q + \mathcal{B}_p(\mathcal{H})$ *is a real analytic submanifold if and only if the spectrum of* B^*B *is finite.*

Proof. If $Q = UQ_0U^* = Ad(U)(Q)$, then $Ad(U)$ is an isometric isomorphism between $Q + \mathcal{B}_p(\mathcal{H})$ and $Q_0 + \mathcal{B}_p(\mathcal{H})$ which sends $\mathcal{U}_{p,Q}$ to $\mathcal{U}_{p,Q_0}$, and it is also a $*$-ismorphism between $\mathcal{C}^*(Q)$ and $\mathcal{C}^*(Q_0)$. Therefore we may reason with Q_0. Sufficiency was proved in the previous propostion. Let us suppose that $\mathcal{U}_{p,Q_0} \subset Q_0 + \mathcal{B}_p(\mathcal{H})$ is a real analytic submanifold, and prove that the spectrum of B_0 must be finite. This assumption implies that the differential $\delta_{Q_0} = (d\pi_{Q_0})_1$,

$$\delta_{Q_0} : \mathcal{B}_p(\mathcal{H})_{ah} \to \mathcal{B}_p(\mathcal{H})$$

has complemented range. In particular, it has closed range. Let $Y(n)$, $n \geq 1$ be a sequence in $\mathcal{B}_p(\mathcal{H})_{ah}$. Note that

$$\delta_{Q_0}(Y(n)) = \begin{pmatrix} B_0Y(n)^*_{1,2} & Y(n)_{1,2} + Y(n)_{1,1}B_0 - B_0Y(n)_{2,2} \\ -Y(n)^*_{1,2} & -Y(n)^*_{1,2}B_0 \end{pmatrix}.$$

Therefore if $\delta_{Q_0}(Y(n))$ is convergent, then $Y(n)_{1,2}$ is convergent, and thus also

$$Y(n)_{1,1}B_0 - B_0Y(n)_{2,2}$$

is convergent. These facts imply that if δ_{Q_0} has closed range, then also the restriction

$$\{\begin{pmatrix} X_{1,1} & 0 \\ 0 & X_{2,2} \end{pmatrix} : X^*_{i,i} = -X_{i,i}\} \xrightarrow{\delta_{Q_0}} \mathcal{B}_p(\mathcal{H})$$

has closed range. One may further restrict δ_{Q_0} to

$$\{\begin{pmatrix} X_0 & 0 \\ 0 & X_0 \end{pmatrix} : X^*_0 = -X_0\},$$

and it will still have closed range. Indeed, suppose that $X_nB_0 - B_0X_n \to Y$ for X_n in $\mathcal{B}_p(\mathcal{H}_0)_{ah}$. The closed range condition implies that there exist $X_0, Z_0 \in \mathcal{B}_p(\mathcal{H}_0)_{ah}$ such that $Y = X_0B_0 - B_0Z_0$. Thus, taking adjoints and substracting, one has $0 = (X_0 - Z_0)B_0 - B_0(X_0 - Z_0)$. Therefore

$$Y = X_0B_0 - B_0Z_0 - \frac{1}{2}(X_0 - Z_0)B_0 + \frac{1}{2}B_0(X_0 - Z_0) = \frac{X_0 + Z_0}{2}B_0 - B_0\frac{X_0 + Z_0}{2}.$$

Thus we have reduced the situation to the following problem self-adjoint case: prove that if

$$\delta_{B_0} : \mathcal{B}_p(\mathcal{H}_0)_{ah} \to \mathcal{B}_p(\mathcal{H}_0)$$

has closed range, then B_0 must have finite spectrum. Therefore we finish the proof if we prove the following Lemma. Note that δ_{B_0} maps $\mathcal{B}_p(\mathcal{H}_0)_{ah}$ into $\mathcal{B}_p(\mathcal{H}_0)_h$ □

Before we establish the referred Lemma, let us transcribe the following result by L. Fialkow [12]: denote by τ_{AB} the operator $\tau_{AB}(X) = AX - XB$. Let $\mathcal{J}$ be any Schatten ideal.

Theorem 6.5. (Fialkow [12]) *The following are equivalent:*

1. $\tau_{AB} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *is bounded below.*

2. $\tau_{AB} : \mathcal{J} \to \mathcal{J}$ *is bounded below for some* $\mathcal{J}$.

3. $\tau_{AB} : \mathcal{J} \to \mathcal{J}$ *is bounded below for any* $\mathcal{J}$.

4. $\sigma_l(A) \cap \sigma_r(B) = \emptyset$.

Here $\sigma_l(A)$ (resp. $\sigma_r(B)$) denote the left (resp. right) spectrum of A (resp. B). In our particular case, we deal with $\tau_{AA} = \delta_A$ and $\mathcal{J} = \mathcal{K}(\mathcal{H})$.

Lemma 6.6. *Let $A^* = A \in \mathcal{B}(\mathcal{H})$. If the map $\delta_A : \mathcal{B}_p(\mathcal{H})_{ah} \to \mathcal{B}_p(\mathcal{H})_h$ has closed range then the spectrum of A is finite.*

Proof. Denote by $\delta_A^{\mathbb{C}}$ the map $\delta_A^{\mathbb{C}} : \mathcal{B}_p(\mathcal{H}) \to \mathcal{B}_p(\mathcal{H})$ defined accordingly. Clearly

$$\mathcal{B}_p(\mathcal{H}) = \mathcal{B}_p(\mathcal{H})_h \oplus \mathcal{B}_p(\mathcal{H})_{ah},$$

$$\delta_A^{\mathbb{C}}(\mathcal{B}_p(\mathcal{H})_h) \subset \mathcal{B}_p(\mathcal{H})_{ah} \quad \text{and} \quad \delta_A^{\mathbb{C}}(\mathcal{B}_p(\mathcal{H})_{ah}) \subset \mathcal{B}_p(\mathcal{H})_h.$$

Therefore it is apparent that δ_A has closed range if and only if $\delta_A^{\mathbb{C}}$ does. Let us denote this latter map also by δ_A to lighten the notation.

The Hilbert space $\mathcal{H}$ can be decomposed in two orthogonal subspaces $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}$ which reduce A, such that $A_c = A|_{\mathcal{H}_c} \in \mathcal{B}(\mathcal{H}_c)$ has continuous spectrum, and the spectrum of $A_{pp} = A|_{\mathcal{H}_{pp}}$ has a dense subset of eigenvectors. We claim that δ_{A_c} and $\delta_{A_{pp}}$ have both closed range. Suppose $x_n \in \mathcal{B}(\mathcal{H}_c)$ is such that $\delta_{A_c}(X_n) \to Y$, then $Y_n = X_n \oplus 0 \in \mathcal{B}(\mathcal{H})$ satisfy

$$\delta_A(Y_n) = \delta_{A_c}(X_n) \oplus 0 \to Y \oplus 0$$

in $\mathcal{B}(\mathcal{H})$, and thus $Y \oplus 0 = \delta_A(X)$. If one writes this equality in matrix form (in terms of the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}$), one has

$$\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}A_c - A_cX_{11} & X_{12}A_{pp} - A_cX_{12} \\ X_{21}A_c - A_{pp}X_{21} & X_{22}A_{pp} - A_{pp}X_{22} \end{pmatrix},$$

and therefore $Y = \delta_{A_c}(X_{11})$. Analogously one proves that the range of $\delta_{A_{pp}}$ is closed. In order to prove that the spectrum of A is finite, we must show first that $\mathcal{H}_c$ is trivial. First note that $\delta_{A_c} : \mathcal{B}_p(\mathcal{H}_c) \to \mathcal{B}_p(\mathcal{H}_c)$ has trivial kernel. Indeed, if $X \neq 0$ is a compact operator commuting with A_c, since also $X + X^*$ commutes with A_c, by the spectral decomposition of compact self-adjoint operators, one can find a non trivial (finite rank) spectral projection of $X + X^*$, which commutes with A_c, and thus A_c would have an eigenvalue, leading to a contradiction. It follows that $\delta_{A_c} : \mathcal{B}_p(\mathcal{H}_c) \to \mathcal{B}_p(\mathcal{H}_c)$ is bounded from below. Thus, by Fialkow's theorem above, one would have that $\sigma_l(A_c) \cap \sigma_r(A_c) = \emptyset$. Since for self-adjoint operators, right and left spectra coincide, this implies that the spectrum of A_c is empty, and therefore $\mathcal{H}_c$ is trivial.

It follows that the spectrum of A has a dense subset of eigenvalues. Suppose that there are infinitely many eigenvalues. By adding a multiple of the identity to A (a change that does not affect δ_A), we may suppose that 0 is a accumulation point of the set of eigenvalues of A. From this infinite set one can select a sequence of (different) eigenvalues $\{\lambda_n : n \geq 1\}$ which are square summable. For each $n \geq 1$ pick a unit eigenvector e_n, consider $\mathcal{H}_0$ the closed linear span of these eigenvectors, and denote $A_0 = A|_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H}_0)$. Note that A_0 is a Hilbert-Schmidt operator. It is apparent that since $\delta_A : \mathcal{B}_p(\mathcal{H}) \to \mathcal{B}_p(\mathcal{H})$ has closed range, then $\delta_{A_0} : \mathcal{B}_p(\mathcal{H}_0) \to \mathcal{B}_p(\mathcal{H}_0)$ also has closed range. Let us show that the kernel is complemented. Note that A_0, written in the orthogonal basis $\{e_n : n \geq 1\}$ is a diagonal infinite matrix, with different entries in the diagonal. Thus the kernel of δ_{A_0}, which is formed by the p-summable operators in $\mathcal{H}_0$ which commute with A_0, consists also of diagonal matrices. Therefore $N(\delta_{A_0})$ is complemented, and one can choose the projection P onto $N(\delta_{A_0})$ given by

$$\Pi : \begin{pmatrix} X_{11} & X_{12} & X_{13} & \dots \\ X_{21} & X_{22} & X_{23} & \dots \\ X_{31} & X_{32} & X_{33} & \dots \\ \dots \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & 0 & 0 & \dots \\ 0 & X_{22} & 0 & \dots \\ 0 & 0 & X_{33} & \dots \\ \dots \end{pmatrix}.$$

Then $\mathcal{B}_p(\mathcal{H}_0) = N(\delta_{A_0}) \oplus L$, with $L = N(P)$, and

$$\delta_{A_0}|_L : L \to R(\delta_{A_0})$$

is an isomorphism between Banach spaces. It follows that there exists a constant $C > 0$ such that

$$\|XA_0 - A_0X\|_p \geq C\|X - \Pi(X)\|_p, \quad \text{for all } X \in \mathcal{B}_p(\mathcal{H}_0).$$

For each $k \geq 1$, consider the $k \times k$ matrix b_k with $\frac{1}{k}$ in all entries, and let X_k be the operator in $\mathcal{H}_0$ whose matrix has b_k in the first $k \times k$ corner and zero elsewhere. Note that X_k is a rank one orthogonal projection and thus $\|X_k\|_p = 1$. Also note that $\|\Pi(X_k)\|_p = \frac{1}{k^{p-1}}$. It follows that

$$\|X_k - \Pi(X_k)\|_p \to 1$$

as $k \to \infty$. On the other hand, $X_kA_0 - A_0X_k$ is a Hilbert-Schmidt operator whose 2-norm squared is

$$\begin{aligned} \|X_kA_0 - A_0X_k\|_2^2 &= \frac{1}{k^2}\sum_{i,j=1}^{k} \lambda_j^2 + \lambda_i^2 - 2\lambda_j\lambda_i = \frac{2}{k^2}\{k\sum_{i=1}^{k}\lambda_i^2 - (\sum_{i=1}^{k}\lambda_i)^2\} \\ &\leq \frac{2}{k}\sum_{i=1}^{k}\lambda_i^2 \leq \frac{2}{k}\|A_0\|_2^2. \end{aligned}$$

Thus $\|X_kA_0 - A_0X_k\|_p \leq \|x_kA_0 - A_0x_k\|_2 \to 0$, leading to a contradiction. It follows that the spectrum of A is finite. □

7 Finsler metrics in the p-Schatten unitary orbits

Let us describe a natural Finsler metric for homogeneous spaces of the p-Schatten unitary groups, and the minimality results obtained for it [2], for p and even integer. The case $p = \infty$, i.e. the case of the orbit under the action of the Fredholm group, is similar, though it has some differences and is treated apart. These results in [2] apply to the particular case of the unitary orbit of Q. The metric is induced by the action of $\mathcal{U}_p(\mathcal{H})$ and the p-norm, and is formally the same metric considered for the full orbit of Q. We should mention that it is not required here that $\mathcal{U}_{p,Q}$ be a submanifold of $\mathcal{B}_p(\mathcal{H})$, the method applies if $\mathcal{U}_{p,Q}$ has smooth structure which makes the map

$$\pi_Q : \mathcal{U}_p(\mathcal{H}) \to \mathcal{U}_{p,Q}, \quad \pi_Q(U) = UQU^*$$

a smooth submersion. Using the same argument as in Proposition 3.4, one can prove that this is always the case. If $R \in \mathcal{U}_{p,Q}$ and $v = ZR - RZ \in (T\mathcal{U}_{p,Q})$, put

$$\|v\|_R = \inf\{\|Z\|_p : Z \in \mathcal{B}_p(\mathcal{H})_{ah},\ ZR - RZ = v\}.$$

The metric is invariant under the action of the group. The second important fact of this metric is that, due to the uniform comvexity of the p-norm, there is a unique minimal lifting Z_0, $v = Z_0 R_R Z_0$, achieving the infimum,

$$\|Z_0\|_p = \|v\|_R.$$

This situation contrasts with the operator norm Finsler metric, where multiple minimal liftings can occur. If $p = 2$, it can be shown that the minimal lifting lies in the reductive space $\mathcal{K}_R$. If $p > 2$, the set of minimal liftings is, in general, not a linear subspace. The main result from [2], adapted to this context, is the following:

Theorem 7.1. *If $R \in \mathcal{U}_{p,Q}$ and Z_0 is the minimal lifting for $v \in (T\mathcal{U}_{p,Q})_R$, then the curve*

$$\delta(t) = e^{tZ_0} R e^{-tZ_0}$$

which starts at R with initial velocity v has minimal length for $|t| < \frac{\pi}{4\|Z_0\|_p}$. Moreover, δ is unique with this property.

If we denote by d_p the metric in $\mathcal{U}_{p,Q}$ induced by this Finsler metric, also from [2] we have the following:

Theorem 7.2. *The metric space $(\mathcal{U}_{p,Q}, d_p)$ is complete.*

References

[1] E. Andruchow, G. Corach, Differential geometry of partial isometries and partial unitaries, Illinois Journal of Mathematics 48, No. 1 (2004), 97-120

[2] E. Andruchow, G. Larotonda, L. Recht Finsler geometry and actions of the p-Schatten unitary groups, Trans. Amer. Math. Soc. (to appear).

[3] E. Andruchow, L. Mata-Lorenzo, A. Mendoza ,A. Varela, Minimal Matrices and the Corresponding Minimal Curves on Flag Manifolds in Low Dimension, Lin. Alg. Appl. (to appear)

[4] E. Andruchow, D. Stojanoff, Geometry of unitary orbits, J. Operator Theory 26 (1991), no. 1, 25-41.

[5] D. Beltita, Smooth Homogeneous Structures in Operator Theory, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 137, Chapman & Hall/CRC Press, Boca Raton-London-New York-Singapore, 2006.

[6] G. Corach, H. Porta, L. Recht, The geometry of spaces of projections in C^*-algebras, Adv. Math. 101 (1993), 59–77.

[7] C. Davis, W. M. Kahan, The rotation of eigenvectors by a perturbation. III, SIAM J. Numer. Anal. 7 (1970), 1–46.

[8] J. Dixmier, Position relative de deux variétés linéaires fermées dans un espace de Hilbert, Revue Sci. 86 (1948), 387–399.

[9] J. Dixmier, Étude sur les variétés et les opérateurs de Julia, avec quelques applications, Bull. Soc. Math. France 77 (1949). 11–101.

[10] D. Z. Dokovic. Unitary similarity of projectors. , Aequationes Math., 42(1991), 220-224.

[11] C.E. Durán, L.E. Mata-Lorenzo, L. Recht, Metric geometry in homogeneous spaces of the unitary group of a $C\Sigma_p*$-algebra. I. Minimal curves, Adv. Math. 184 (2004), 342–366.

[12] L. A. Fialkow, A note on norm ideals and the operator $X \longrightarrow AX - XB$. Israel J. Math. 32 (1979), 331-348.

[13] P. R. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144 (1969), 381-389.

[14] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York - London, 1962.

[15] Kh. D. Ikramov, Canonical forms of projectors with respect to unitary similarity and their applications, Comput. Math. Math. Phys. 44 (2004), 1456–1461.

[16] N. Krupnik, S. Roch, B. Silbermann, On $C*$-algebras generated by idempotents J. Funct. Anal. 137 (1996), 303–319.

[17] L.E. Mata-Lorenzo,L. Recht, Infinite-dimensional homogeneous reductive spaces, Acta Cient. Venezolana 43 (1992), 76–90.

[18] G.K. Pedersen, Measure theory for C^* algebras. II, Math. Scand. 22 (1968), 63–74.

[19] H. Porta, L. Recht, Minimality of geodesics in Grassmann manifolds, Proc. Amer. Math. Soc. 100 (1987), 464–466.

[20] I. Raeburn, The relationship between a commutative Banach algebra and its maximal ideal space, J. Funct. Anal. 25 (1977), 366-390.

[21] I. Raeburn, A. M. Sinclair, The $C\Sigma_p*$-algebra generated by two projections Math. Scand. 65 (1989), no. 2, 278–290.

Instituto Argentino de Matemática, CONICET,
Saavedra 15, (1083) Buenos Aires, Argentina
eandruch@ungs.edu.ar, gcorach@fi.uba.ar, amaestri@fi.uba.ar