DIFFERENTIAL GEOMETRY OF PARTIAL ISOMETRIES AND PARTIAL UNITARIES*

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Abstract

Let $\mathcal{A}$ be a C*-algebra. In this paper the sets $\mathcal{I}$ of partial isometries and $\mathcal{I}_\Delta \subset \mathcal{I}$ of partial unitaries (= partial isometries which commute with their adjoints) are studied from a differential geometric point of view. These sets are complemented submanifolds of $\mathcal{A}$. Special attention is paid to geodesic curves. The space $\mathcal{I}$ is a homogeneous reductive space of the group $U_\mathcal{A} \times U_\mathcal{A}$, where $U_\mathcal{A}$ denotes the unitary group of $\mathcal{A}$, and geodesics are computed in a standard fashion. Here we study the problem of existence and uniqueness of geodesics joining two given endpoints. The space $\mathcal{I}_\Delta$ is *not* homogeneous, and therefore a completely different treatment is given. A principal bundle with base space $\mathcal{I}_\Delta$ is introduced, and a natural connection in it defined. Additional data, namely certain translating maps, enable one to produce a *linear* connection in $\mathcal{I}_\Delta$, whose geodesics are characterized.

Keywords: partial isometry, projection.

1 Introduction

In their study of the problem of unitary equivalence of operators on a Hilbert space H, Halmos and Mc Laughlin [22] proved that the problem can be reduced to that of the unitary equivalence of partial isometries. In doing so, they characterized the connected components of the set $\mathcal{I}$ of all the partial isometries on H: the partial isometries x and y belong to the same component if and only if they have the same nullity (i.e. dimension of the null-space), the same rank (dimension of the image) and the same corank (dimension of the orthogonal complement of the image). They also proved that if $\|x - y\| < 1$ then there exist unitary operators u and v on H such that $y = uxv^*$. This paper is devoted to the study of the differential geometry of the set $\mathcal{I}$. In order to describe the results, we fix a unital C*-algebra $\mathcal{A}$, and denote by $G_\mathcal{A}$ the group of invertible elements of $\mathcal{A}$, by $U_\mathcal{A}$ the subgroup of

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unitary elements of $\mathcal{A}$ and by $\mathcal{P}$ the set of all hermitian projections of $\mathcal{A}$: $\mathcal{P} = \{p \in \mathcal{A} : p^2 = p = p^*\}$. The set $\mathcal{I}$ of partial isometries of $\mathcal{A}$ is defined by $\mathcal{I} = \{x \in \mathcal{A} : x^*x, xx^* \in \mathcal{P}\}$. The differential geometry of $\mathcal{P}$ is well known by now, and we often use this knowledge in order to obtain results on $\mathcal{I}$. The main link between $\mathcal{I}$ and $\mathcal{P}$ is provided by the mapping

$$\mathcal{I} \to \mathcal{P} \times \mathcal{P}, \quad x \mapsto (xx^*, x^*x).$$

Recall that xx^* (resp. x^*x) is called the final (resp. initial) projection of x.

We shall reformulate Halmos and Mc Laughlin's result in geometrical terms. Observe that the map

$$U_{\mathcal{A}} \times U_{\mathcal{A}} \times \mathcal{I} \to \mathcal{I}, \quad (u, v, x) \mapsto uxv^*$$

defines a left action of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ on $\mathcal{I}$. Their result says that the action is locally transitive, i.e. two partial isometries which are close enough are conjugate by a pair of unitaries. As a corollary, the connected component of x in $\mathcal{I}$ is the orbit of x by the action of $U_{\mathcal{A}} \times U_{\mathcal{A}}$. Moreover, it is a homogeneous space of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ and a C^∞ submanifold of $\mathcal{A} = \mathcal{B}(H)$. For a general C*-algebra $\mathcal{A}$, for which $U_{\mathcal{A}}$ is not necessarilly connected, the same argument can be done replacing $U_{\mathcal{A}}$ by U_0, the connected component of 1 in $U_{\mathcal{A}}$. The fact that the connected components of $\mathcal{I}$ are homogeneous spaces of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ but not of $U_{\mathcal{A}}$ depends on the map $\mathcal{I} \to \mathcal{P} \times \mathcal{P}$ mentioned above. In fact, $U_{\mathcal{A}}$ has a left action on $\mathcal{P}$ by $(u, p) \mapsto upu^*$, and the action is locally transitive (see B. Sz.-Nagy [37]): if $p, q \in \mathcal{P}$ and $\|p - q\| < 1$ there exists $u \in U_{\mathcal{A}}$ such that $q = upu^*$ (originally proved for $\mathcal{A} = \mathcal{B}(H)$, true for any C*-algebra, where u can be chosen in U_0). So that the orbit of p by the action on U_0 is the connected component of p in $\mathcal{P}$. Now, the motion of $x \in \mathcal{I}$ is ruled by the motions of its initial and final projections (spaces) x^*x and xx^* in $\mathcal{P}$, and these projections are moved independently. Therefore it must be $U_{\mathcal{A}} \times U_{\mathcal{A}}$ who provides the motions of $x \in \mathcal{I}$ (if one wants the action to be locally transitive and to fill connected components).

For a fixed $p \in \mathcal{P}$, an important role is played by the space $\mathcal{I}^p$ of all partial isometries with initial projection p: $\mathcal{I}^p = \{v \in \mathcal{A} : v^*v = p\}$. $\mathcal{I}^p$ is a submanifold of $\mathcal{I}$, and naturally carries a left action of $U_{\mathcal{A}}$ which makes it a homogeneous space of $U_{\mathcal{A}}$.

The present paper is an addition to the existing literature on differential geometry of different sets and maps of operators (and their correspondent abstract analogues). The reader is referred to [36], [13], [3], [27], [38], [11], [23], [24] for projections, [12], [27], [28] for n-tuples of projections, [7] for spectral measures, [8], [9] for nilpotent operators, [15] for selfadjoint invertible operators, [16], [17], [19], [20] for positive (hermitian) operators, [2], [18] for representations of groups and algebras, [10] for states, [33], [34] for partial isometries, [21], [14], [12] for elements which admit generalized inverses.

Let us describe now the contents of the paper. In section 2 we introduce a linear connection in $\mathcal{I}$, of which we compute the geodesic curves, and investigate how the geometry of $\mathcal{P}$ and $\mathcal{I}^p$ play a role in the geometric properties of $\mathcal{I}$. The general theory of reductive spaces guarantees the existence of a (uniform) radius R, with the property that if two elements lie at distance (measured with the norm of $\mathcal{A}$) which is less than R, then they can be joined by a unique geodesic of this connection. For instance, for the space $\mathcal{P}$ of projections of $\mathcal{A}$ this $R_{\mathcal{P}}$ is 1 [36]. In section 3 we estimate (very roughly) the geodesic radius for the spaces $\mathcal{I}^p$ and for $\mathcal{I}$. This estimation is given relating the three numbers $R_{\mathcal{P}} = 1$, $R_{\mathcal{I}^p}$ and $R_{\mathcal{I}}$.

The second half of this paper (sections 4 and 5) is devoted to the study of what we call *partial unitaries* (there might be another name for them in the literature that we are unaware of): namely, partial isometries such that the initial and final spaces coincide. Or, equivalently, elements $v \in \mathcal{I}$ such that v commutes with v^*. We denote the set of all such by $\mathcal{I}_\Delta$. We show that $\mathcal{I}_\Delta$ is a complemented submanifold of $\mathcal{I}$ (and of $\mathcal{A}$). However, $\mathcal{I}_\Delta$ does not admit a (locally transitive) action of a group of unitaries, as $\mathcal{I}$ does, and it is not a homogeneous space. Therefore, the differential geometric study of $\mathcal{I}_\Delta$ is more complicated. First, note that if $w_1,\ w_2 \in \mathcal{I}_\Delta$ lie at distance which is less than 1, their initial (=final) projections are unitarily equivalent. This motivates the study of the following map. For each fixed $p \in \mathcal{P}$,

$$\pi_p^\Delta : \Delta^p = \Delta := \{(\alpha,\beta) \in U_\mathcal{A} \times U_\mathcal{A} : \alpha p \alpha^* = \beta p \beta^*\} \to \mathcal{I}_\Delta, \ \pi_p^\Delta(\alpha,\beta) = \alpha p \beta^*.$$

If w_1 is in the range of this map, and $w_2 \in \mathcal{I}_\Delta$ satisfies $\|w_2 - w_1\| < 1$, then w_2 also lies in the range of π_p^Δ. In other words, the range of this map fills connected components.

We show that this map is a smooth principal bundle, with structure group

$$G_p = \{(g_1,g_2) \in U_\mathcal{A} \times U_\mathcal{A} : g_i p = p g_i, \ i=1,2 \text{ and } g_1 p = g_2 p\}.$$

We introduce a connection on this principal bundle, and compute the horizontal lifting differential equations. These enable one to perform the parallel transport of elements in the fibres. However our interest is in a connection in the tangent bundle. Unfortunately, this cannot be done as is custom in differential geometry, because the structure group G_p does not act on the tangent spaces of $\mathcal{I}_\Delta$. We introduce a linear connection by means of a distribution of isomorphisms between the horizontal spaces. Namely, if $(\alpha,\beta),(\delta,\epsilon) \in \Delta$, and $\mathcal{K}_{(\alpha,\beta)}$, $\mathcal{K}_{(\delta,\epsilon)}$ denote the corresponding horizontal subspaces, we define a smooth distribution of (real) linear isomorphisms

$$T_{\alpha,\beta}^{\delta,\epsilon} : \mathcal{K}_{(\alpha,\beta)} \to \mathcal{K}_{(\delta,\epsilon)}$$

with the following properties:

1. $T_{\alpha,\beta}^{\alpha,\beta} = id$
2. $(T_{\alpha,\beta}^{\delta,\epsilon})^{-1} = T_{\delta,\epsilon}^{\alpha,\beta}$
3. The distribution is equivariant with respect to the action of G_p: if $(g_1,g_2) \in G_p$ and $(x_1,x_2) \in \mathcal{K}_{(\alpha,\beta)}$,
$$\left(T_{\alpha,\beta}^{\delta,\epsilon}(x_1,x_2)\right) \cdot (g_1,g_2) = T_{\alpha g_1,\beta g_2}^{\delta g_1,\epsilon g_2}(x_1 g_1, x_2 g_2).$$

These maps, combined with the horizontal liftings, provide a parallel transport of tangent vectors in $T\mathcal{I}_\Delta$, and therefore a linear connection. The geodesics of this connection are not explicitely computed, however we show that their (horizontal) liftings satisfy a linear differential equation. This implies in particular that geodesics of this connection in $\mathcal{I}_\Delta$ exist for all $t \in \mathbb{R}$.

2 The reductive structure of $\mathcal{I}$

The group $U_\mathcal{A} \times U_\mathcal{A}$ acts on $\mathcal{I}$ by means of

$$(u,w) \cdot v = u v w^*, \ u,w \in U_\mathcal{A}, \ v \in \mathcal{I}.$$

The action is locally transitive, two partial isometries at distance less than 1 are conjugate by this action, with a pair of unitaries which can be chosen as an explicit (and smooth) formula, which gives local cross sections for the action. Indeed, it is well known [22] that if $v_0, v \in \mathcal{I}$ such that $\|v_0 - v\| < 1$ then $\|v_0 v_0^* - v v^*\| < 1$ and $\|v_0^* v_0 - v^* v\| < 1$. Then [36],[35] there exist unitaries $\nu, \sigma \in U_\mathcal{A}$, which are smooth functions of v_0, v, such that $\nu v_0^* v_0 \nu^* = v^* v$ and $\sigma v_0 v_0^* \sigma^* = v v^*$. Put $\gamma = v \nu^* u_0^* + \sigma(1 - u_0 u_0^*)$; then the pair of unitaries (γ, ν) satisfies

$$\gamma v_0 \nu^* = v.$$

In other terms, the map

$$\mu_{v_0} : \{v \in \mathcal{I} : \|v - v_0\| < 1\} \to U_\mathcal{A} \times U_\mathcal{A}, \ \mu_{v_0}(v) = (\gamma, \nu)$$

is a C^∞ cross section for the action.

Let us fix a partial isometry v_0. We will describe the isotropy subgroup and the tangent spaces based on v_0. The isotropy subgroup $V_{v_0} \subset U_{\mathcal{A}} \times U_{\mathcal{A}}$ is

$$V_{v_0} = \{(f,g) \in U_{\mathcal{A}} \times U_{\mathcal{A}} : fv_0 = v_0 g\}.$$

Note that if $(f,g) \in V_{v_0}$ then f commutes with the final projection $p_0 = v_0 v_0^*$, and g commutes with the initial projection $q_0 = v_0^* v_0$.

Let π_{v_0} be the map (in fact, C^∞ fibre bundle [1])

$$\pi_{v_0} : U_{\mathcal{A}} \times U_{\mathcal{A}} \to \mathcal{I},\ \pi_{v_0}(u,w) = uv_0 w^*.$$

The tangent space of $U_{\mathcal{A}} \times U_{\mathcal{A}}$ at $(1,1)$ identifies with $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$, where $\mathcal{A}_{ah}$ is the real Banach space of antihermitic elements of $\mathcal{A}$. Let $\delta_{v_0} = d(\pi_{v_0})_{(1,1)}$, namely

$$\delta_{v_0} : \mathcal{A}_{ah} \times \mathcal{A}_{ah} \to T\mathcal{I}_{v_0},\ \delta_{v_0}(x,y) = xv_0 - v_0 y.$$

The tangent space (and Lie algebra) $(TV_{v_0})_{(1,1)} = \mathcal{V}_{v_0}$ equals then the kernel of δ_{v_0}, and because π_{v_0} is a submersion, $T\mathcal{I}_{v_0}$ is the image of δ_{v_0}. We shall introduce a reductive structure on $\mathcal{I}$, which means a (real) closed linear subspace $\mathcal{H}_{v_0}$ of $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$ with the following properties:

1. $\mathcal{H}_{v_0} \oplus \mathcal{V}_{v_0} = \mathcal{A}_{ah} \times \mathcal{A}_{ah}$.
2. $ad(f,g)(\mathcal{H}_{v_0}) = \mathcal{H}_{v_0}$.

Here $ad(f,g)(x,y) = (fxf^*, gyg^*)$.

Consider the following linear map

$$\Sigma_{v_0} : \mathcal{A} \to \mathcal{A} \times \mathcal{A},\ \Sigma_{v_0}(a) = (av_0^* - v_0 a^* + v_0 a^* v_0 v_0^*, a^* v_0 - v_0^* a + 2v_0^* a v_0 v_0^*).$$

The following result is a straightforward computation.

Lemma 2.1 $\delta_{v_0} \circ \Sigma_{v_0} \circ \delta_{v_0} = \delta_{v_0}$.

Note that in particular, the range of $\Sigma_{v_0} \circ \delta_{v_0}$ lies in $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$. This map $\Sigma_{v_0} \circ \delta_{v_0}$ is therefore an idempotent in the Banach algebra $B_{\mathbb{R}}(\mathcal{A}_{ah} \times \mathcal{A}_{ah})$ of real linear bounded operators on the space $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$, whose kernel equals the kernel of $\delta_{v_0} = \mathcal{V}_{v_0}$. Therefore, its range is a supplement of $\mathcal{V}_{v_0}$ in $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$. Let us define

$$\mathcal{H}_{v_0} := R(\Sigma_{v_0} \circ \delta_{v_0}) = R(\Sigma_{v_0}|_{(T\mathcal{I})_{v_0}}).$$

Explicitely,

$$\mathcal{H}_{v_0} = \{(xp_0 + p_0 x - p_0 x p_0 - v_0 y v_0^*, yq_0 + q_0 y - 2q_0 y q_0) \in \mathcal{A}_{ah} \times \mathcal{A}_{ah} : x, y \in \mathcal{A}_{ah}\}.$$

We claim that it is an invariant supplement of $\mathcal{V}_{v_0}$.

Lemma 2.2 *If $(f,g) \in V_{v_0}$, and $(x,y) \in \mathcal{H}_{v_0}$, then*

$$ad(f,g)(x,y) = (fxf^*, gyg^*) \in \mathcal{H}_{v_0}.$$

Proof. The element (x,y) is of the form $\Sigma_p(av_0 - v_0 b)$ for $a, b \in \mathcal{A}_{ah}$, i.e.

$$x = ap_0 + p_0 a - p_0 b p_0 - v_0 a v_0^*, \quad \text{and } y = q_0 b + bq_0 - 2q_0 b q_0.$$

Then

$$fxf^* = fap_0 f^* + fp_0 af^* - fp_0 ap_0 f^* - fv_0 bv_0^* f^* = faf^* p_0 + p_0 faf^* - p_0 faf^* p_0 - v_0 gbg^* v_0^*,$$

where in the last term we use that $fv_0 = v_0 g$. Analogously

$$fyf^* = q_0 fbf^* + fbf^* q_0 - 2q_0 fbf^* q_0.$$

Then $(fxf^*, gyg^*) = \Sigma_{v_0} \circ \delta_{v_0}(faf^*, fbf^*)$ which lies in $\mathcal{H}_{v_0}$ because $faf^*, gbg^* \in \mathcal{A}_{ah}$. $\square$

This reductive structure induces a linear connection in $\mathcal{I}$. We are interested in the exponential map and the geodesics curves of this connection. These can be computed in a standard fashion [25] (see [32] for a C*-algebraic framework). For instance, if $x \in T\mathcal{I}_{v_0}$ then the unique geodesic $\gamma(t) \in \mathcal{I}$ with $\gamma(0) = v_0$ and $\dot{\gamma}(0) = x$ is

$$\gamma(t) = e^{t\xi_1} v_0 e^{-t\xi_2},$$

where $\xi = (\xi_1, \xi_2) = \Sigma_p(x)$. Geodesics starting at an arbitrary point $v = uv_0w^* \in \mathcal{I}$ are transports of geodesics starting at v_0: $\nu(t) = u\gamma(t)w^*$ with γ as above. Note that the action defines linear isomorphisms between the correspondent tangent spaces. It follows that (as is usual with homogeneous spaces) the local structure of $\mathcal{I}$ can be studied on the neighbourhoods of v_0.

The projections p_0 and q_0 enable one to regard the elements of $\mathcal{A}$ as 2×2 matrices. Let us describe the matrices of horizontal elements $\xi \in \mathcal{H}_{v_0}$. If $\xi = (\xi_1, \xi_2) = (ap_0 + p_0a - p_0xp_0 - v_0yv_0^*, q_0y + yq_0 - 2q_0yq_0)$, $y^* = -y$, $x^* = -x$, then it is straightforward that ξ is of the form

$$\xi_1 = \begin{pmatrix} x_{11} & x_{12} \\ -x_{12}^* & 0 \end{pmatrix}_{p_0}, \ \xi_2 = \begin{pmatrix} 0 & y_{12} \\ -y_{12}^* & 0 \end{pmatrix}_{q_0}$$

with $x_{11}^* = -x_{11}$. The subindices p_0,q_0 mean that the matrices are regarded with respect to these projections.

We finish this section by recalling the connections of the homogeneous spaces $\mathcal{P}$ [13] and $\mathcal{I}^{q_0}$ (denoted $\mathcal{S}_{q_0}(\mathcal{A})$ in) [6]. This recollection will make apparent the close relationship between these geometries.

Theorem 2.3 *Geodesics of $\mathcal{P}$, starting at q_0, are of the form $\rho(t) = e^{t\xi_2} q_0 e^{-t\xi_2}$, where ξ_2 is as the second coordinate of ξ above.*

Geodesics of $\mathcal{I}^{q_0}$, starting at v_0, are of the form $\sigma(t) = e^{t\xi_1} v_0$. where ξ_1 is as the first coordinate of ξ above.

3 Existence of geodesics joining two given endpoints

In [13] (see also [35]) it is shown that two projections $p, q_0 \in \mathcal{P}$ such that $\|p - q_0\| < 1$ can be joined by a unique geodesic. In [4] there is no estimation of the geodesic radius of $\mathcal{I}^{q_0}$. The general theory stipulates the existence of a number $R > 0$ with the property that two elements $v', v'' \in \mathcal{I}^{q_0}$ such that $\|v' - v''\| < R$ can be joined by a unique geodesic. In this section we relate this constant R with the geodesic radius of $\mathcal{I}$, and also give a rough estimate of R.

Let $v, v_0 \in \mathcal{I}$ two partial isometries. We want to establish the existence of a geodesic curve joining v and v_0.

Let us denote by $\mathcal{H}'_{v_0}, \mathcal{H}''_{v_0}$ the subspaces of first and second coordinates of elements of $\mathcal{H}_{v_0}$, i.e.

$$\mathcal{H}_{v_0} = \mathcal{H}'_{v_0} \times \mathcal{H}''_{v_0}.$$

Proposition 3.1 *Suppose that $\|v - v_0\| < \min\{1/2, \frac{R}{3}\}$. Then there exists a unique $\xi = (\xi_1, \xi_2) \in \mathcal{H}_{v_0}$ such that $v = e^{\xi_1} v_0 e^{-\xi_2}$ and $\|\xi_1\| < R$ and $\|\xi_2\| < \pi$.*

Proof. First, note that $\|v^*v - q_0\| \le \|v^*v - v^*q_0\| + \|v^*q_0 - v_0^*v_0\| \le \|v^*\|\|v - v_0\| + \|v^* - v_0^*\| \le 2\|v - v_0\| < 1$. By 2.3, there exists a unique $\xi_2 \in \mathcal{A}_{ah}$, with $\|\xi_2\| < \pi$, $\xi_2 \in \mathcal{H}''_{v_0}$, such that $v^*v = e^{\xi_2} q_0 e^{-\xi_2}$. Let $\hat{v} = ve^{\xi_2}$. Then, it is apparent that $\hat{v}^*\hat{v} = q_0$, i.e. $\hat{v} \in \mathcal{I}^{q_0}$. Compute

$$\|\hat{v} - v_0\| = \|ve^{\xi_2} - v_0\| \le \|ve^{\xi_2} - v\| + \|v - v_0\| < \|e^{\xi_2} - 1\| + \|v - v_0\|.$$

Let us estimate $\|e^{\xi_2}-1\|$. This norm equals $r(e^{\xi_2}-1)$ (r =spectral radius), which is bounded by $\sqrt{2(1-cos(\|\xi_2\|))}$. In [35] (see also [5]) the norm $\|\xi_2\|$ is computed in terms of the projections q_0 and v^*v, namely,

$$\|\xi_2\| = arcsin(\|q_0 - v^*v\|) \leq arcsin(2\|v-v_0\|).$$

Therefore,

$$\|e^{\xi_2}-1\| < \sqrt{2(1-cos(arcsin(2\|v-v_0\|)))} = 2\|v-v_0\|.$$

It follows that

$$\|\hat{v}-v_0\| < 3\|v-v_0\| \leq R.$$

Then there exists a unique $\xi_1 \in \mathcal{H}'_{v_0}$ with $\|\xi_1\| < R$ such that $e^{-\xi_1}v_0 = \hat{v}$. In other words, we have found $\xi = (\xi_1, \xi_2) \in \mathcal{H}_{v_0}$ such that

$$e^{\xi_1}v_0e^{-\xi_2} = \hat{v}e^{-\xi_2} = v.$$

□

Let us give now a rough estimate for R.

Let $\mathcal{V} = \{x \in \mathcal{A}_{ah} : xp_0 = 0\}$. Note that $\mathcal{V}$ is a supplement for $\mathcal{H}'_{v_0}$ in $\mathcal{A}_{ah}$ (it is, in fact, the Lie algebra of vertical elements of the homogeneous space $\mathcal{I}^{q_0}$ [4]).

Our estimate will be based on the map

$$\nu : \mathcal{A}_{ah} \to \mathcal{A}_{ah}, \quad \nu(x) = \log(e^{x_{\mathcal{H}}}e^{x_{\mathcal{V}}}),$$

where

$$x = x_{\mathcal{H}} + x_{\mathcal{V}}$$

is the decomposition of x with $x_{\mathcal{H}} \in \mathcal{H}'_{v_0}$ and $x_{\mathcal{V}} \in \mathcal{V}$, and log is the analytic inverse of the usual exponential of antihermitic elements, on a neighbourhood of $1 \in U_{\mathcal{A}}$. Namely, log is defined on $\{u \in U_{\mathcal{A}} : \|u-1\| < 1\}$, as the series

$$\log(u) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(u-1)^n.$$

Note that since $x_{\mathcal{V}} = (1-q_0)x_{ah}(1-q_0)$, ($x_{ah}$ = the antihermitic part of x), then $\|x_{\mathcal{V}}\| \leq \|x_{ah}\| \leq \|x\|$. However $\|x_{\mathcal{H}}\| \leq 2\|x\|$. This estimate is apparent. Clearly $d\nu_0 = I$. Therefore ν is a local diffeomorphism around the origin. Let $B_r(0)$ denote the ball with centre 0 and radius r in $\mathcal{A}_{ah}$.

Lemma 3.2 *$\nu : B_r(0) \to B_{r/2}(0)$ is a diffeomorphism for $r = 0.036$.*

Proof. Recall the usual proof of inverse function theorem in the context of Banach spaces (see, e.g. [29]). Let $\gamma = id - \nu$, one needs to estimate the norm of the differential of γ at $a \in \mathcal{A}_{ah}$,

$$d\gamma_a(x) = x - d\nu_a(x).$$

Let $b, y \in \mathcal{A}_{ah}$, then $dexp_b(y) = \frac{d}{dt}e^{b+ty}\big|_{t=0}$. This derivative equals the series

$$y + \frac{1}{2}(yb+by) + \frac{1}{6}(yb^2+byb+b^2y) + \ldots$$

Analogously, if $u \in U_{\mathcal{A}}$ with $\|u-1\| < 1$,

$$d\log_u(y) = y - \frac{1}{2}((u-1)x + x(u-1)) + \frac{1}{3}((u-1)^2x + (u-1)x(u-1) + x(u-1)^2 - \ldots$$

A straightforward estimate yields

$$\|d\log_u\| \le \frac{1}{1-\|u-1\|}.$$

Let us denote by $exp(a) = e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}$. Then $d\gamma_a(x) = x - d\log_{exp(a)}(dexp_a(x))$, and

$$\|d\gamma_a(x)\| \le \|x - d\log_{exp(a)}(x)\| + \|d\log_{exp(a)}[x - dexp_a(x)]\|$$

$$\le \|x\|\frac{\|exp(a)-1\|}{1-\|exp(a)-1\|} + \frac{1}{1-\|exp(a)-1\|}\|x - dexp_a(x)\|.$$

Let us estimate $\|x - dexp_a(x)\|$. As above,

$$dexp_a(x) = \frac{d}{dt}e^{a_{\mathcal{H}}+tx_{\mathcal{H}}}e^{a_{\mathcal{V}}+tx_{\mathcal{V}}}\big|_{t=0} = \frac{d}{dt}e^{a_{\mathcal{H}}+tx_{\mathcal{H}}}\big|_{t=0}e^{a_{\mathcal{V}}} + e^{a_{\mathcal{H}}}\frac{d}{dt}e^{a_{\mathcal{V}}+tx_{\mathcal{V}}}\big|_{t=0}.$$

Note that

$$\frac{d}{dt}e^{a_{\mathcal{H}}+tx_{\mathcal{H}}}\big|_{t=0}e^{a_{\mathcal{V}}} = (x_{\mathcal{H}} + \frac{1}{2}(a_{\mathcal{H}}x_{\mathcal{H}} + x_{\mathcal{H}}a_{\mathcal{H}}) + \frac{1}{6}(a_{\mathcal{H}}^2x_{\mathcal{H}} + a_{\mathcal{H}}x_{\mathcal{H}}a_{\mathcal{H}} + x_{\mathcal{H}}a_{\mathcal{H}}^2) + \ldots)e^{a_{\mathcal{V}}}$$

$$= (x_{\mathcal{H}} + R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}}))(1 + (e^{a_{\mathcal{V}}}-1)) = x_{\mathcal{H}} + x_{\mathcal{H}}(e^{a_{\mathcal{V}}}-1) + R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}})e^{a_{\mathcal{V}}}.$$

Analogously,

$$e^{a_{\mathcal{H}}}\frac{d}{dt}e^{a_{\mathcal{V}}+tx_{\mathcal{V}}}|_{t=0} = x_{\mathcal{V}} + (e^{a_{\mathcal{H}}}-1)x_{\mathcal{V}} + e^{a_{\mathcal{H}}}R_{2,a_{\mathcal{V}}}(x_{\mathcal{V}}).$$

Therefore

$$\|x - dexp_a(x)\| \le \|x_{\mathcal{H}}\|\|e^{a_{\mathcal{V}}}-1\| + \|R_{2,a_{\mathcal{H}}}(x_{\mathcal{H}})\| + \|x_{\mathcal{V}}\|\|e^{a_h}-1\| + \|R_{2,a_{\mathcal{V}}}(x_{\mathcal{V}})\|.$$

Note that $\|R_{2,b}(y)\| \le \|y\|(e^{\|b\|}-1)$. Recall the estimates $\|e^z - 1\| \le 2\sin(\|z\|/2)$, $\|z_{\mathcal{V}}\| \le \|z\|$ and $\|z_{\mathcal{H}}\| \le 2\|z\|$ for $z \in \mathcal{A}_{ah}$. Then

$$\|x - dexp_a(x)\| \le 2\|x_{\mathcal{H}}\|\sin(\|a_{\mathcal{V}}\|/2) + 2\|x_{\mathcal{V}}\|\sin(\|a_{\mathcal{H}}\|/2) + \|x_{\mathcal{H}}\|(e^{\|a_{\mathcal{H}}\|}-1) + \|x_{\mathcal{V}}\|(e^{\|a_{\mathcal{V}}\|}-1)$$

$$\le \|x\|\{4\sin(\|a\|/2) + 2\sin(\|a\|) + 2e^{2\|a\|} + e^{\|a\|} - 3\}.$$

We also need to estimate $\|exp(a)-1\|$ in terms of $\|a\|$:

$$\|exp(a)-1\| = \|e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}-1\| \le \|e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}} - e^{a_{\mathcal{V}}}\| + \|e^{a_{\mathcal{V}}}-1\|$$

$$\le 2\sin(\|a\|) + 2\sin(\|a\|/2).$$

It follows that $\|d\gamma_a\|$ is bounded by

$$\frac{1}{1-2\sin(\|a\|)-2\sin(\|a\|/2)}\{6\sin(\|a\|/2) + 4\sin(\|a\|) + 2e^{2\|a\|} + e^{\|a\|} - 3\}.$$

Therefore if $\|a\| < 0.036$ then $\|d\gamma(a)\| < 1/2$. As in the proof of the inverse function theorem [29], this implies that $\nu : B_r(0) \to B_{r/2}(0)$ is a diffeomorphism. □

For $z \in \mathcal{A}_{ah}$ with small norm, such as we are considering, one has in fact the equality

$$\|e^z - 1\| = 3\sin(\|z\|/2).$$

Therefore $z \in B_{r/2}(0)$ if and only if $\|e^z - 1\| < 2\sin(r/4)$. In other words, the above lemma states that

$$exp : B_r(0) \to \{u \in U_{\mathcal{A}} : \|u-1\| < 2\sin(r/4)\}$$

is an analytic diffeomorphism, for $r = 0.036$.

For the uniqueness part we shall need the following result:

Lemma 3.3 *Let $\xi, \xi' \in \mathcal{H}'_{v_0}$ such that $e^{\xi} v_0 = e^{\xi'} v_0$ with $\|\xi\|, \|\xi'\| < d = \ln(5/4)$. Then $\xi = \xi'$.*

Proof. Let $\mu : \mathcal{A} \to \mathcal{A}$ the following (real) linear map

$$\mu(x) = x v_0^* - v_0 x^* (1 - p_0).$$

Clearly $\|\mu(x)\| \leq 2\|x\|$. If $x \in \mathcal{A}$, let $x = x_h + x_{ah}$ be the decomposition of x in its hermitic and antihermitic parts. Let

$$\theta : \mathcal{A} \to \mathcal{A},$$

$$\theta(x) = \mu \left(E \left(x_{ah} - (1 - p_0) x_{ah} (1 - p_0) \right) v_0 \right) + (1 - p_0) x_{ah} (1 - p_0)) + x_h - v_0,$$

where E is the usual exponential. Clearly this map is C^∞. The differential of θ at the origin is the identity. Indeed,

$$d\theta_0(x) = \mu\big(dE_0(x_{ah} - (1 - p_0) x_{ah} (1 - p_0)) v_0\big) + (1 - p_0) x_{ah} (1 - p_0) + x_h$$

$$= \mu(x_{ah} v_0) + (1 - p_0) x_{ah} (1 - p_0) + x_h = x_{ah} p_0 - p_0 x_{ah}^* (1 - p_0) + (1 - p_0) x_{ah} (1 - p_0) + x_h$$

$$= x_{ah} p_0 + x_{ah} (1 - p_0) + x_h = x.$$

It follows that θ is a local diffeomorphism with $\theta(0) = 0$. We proceed as in the previous lemma, considering the auxiliary map $\gamma(x) = x - \theta(x)$ and estimating $\|d\gamma_a\|$. Now

$$d\gamma_a(x) = x - d\theta_a(x) = x_{ah} - (1 - p_0) x_{ah} (1 - p_0) - \mu(dE_{a_{ah}}(x_{ah}) v_0)$$

$$= \mu\big(x_{ah} v_0 - dE_{a_{ah}}(x_{ah}) v_0\big).$$

Therefore

$$\|d\gamma_a(x)\| \leq 2\|x_{ah} - dE_{a_{ah}}(x_{ah})\| \leq 2\|x\|(e^{\|a\|} - 1).$$

Then $\|d\gamma_a\| < 1/2$ if $\|a\| < \ln(5/4)$. As above, this implies that $\theta : B_d(0) \to B_{d/2}(0)$ is a diffeomorphism for $d = \ln(5/4)$. Note then that the map $\theta + v_0$ restricted to $\mathcal{H}'_{v_0} \cap B_d(0)$,

$$\theta + v_0 : \mathcal{H}'_{v_0} \cap B_d(0) \to \mu(E(\mathcal{H}'_{v_0} v_0)), \ \xi \mapsto e^{\xi} v_0$$

is bijective. Therefore if $\xi, \xi' \in \mathcal{H}'_{v_0}$ with $\|\xi\|, \|\xi'\| < d = \ln(5/4)$, satisfy $e^{\xi} v_0 = e^{\xi'} v_0$, then $\mu(e^{\xi} v_0) = \mu(e^{\xi'} v_0)$, which means that $\theta(\xi) = \theta(\xi')$, and then $\xi = \xi'$.

□

Note that $\ln(5/4) \simeq 0.223$ which is bigger than the value of r that we estimated. We may summarize both lemmas in the following:

Corollary 3.4 *Let $r > 0$ such that exp is a diffeomorphism on $B_r(0)$ (with $r \leq d$). If a partial isometry $v \in \mathcal{I}^{q_0}$ satisfies that $\|v - v_0\| < \epsilon = \frac{2}{\sqrt{3}} \sin(r/4)$, then there exists a unique $\xi \in \mathcal{H}'_{v_0}$ such that $v = e^{\xi} v_0$ with $\|\xi\| < 2r$.*

Proof. Suppose that $\|v - v_0\| < \epsilon$. This implies that $\|vv^* - v_0 v_0^*\| < 2\epsilon \leq 1$. It follows that [6] [35] there exists a unitary element $w \in U_{\mathcal{A}}$ such that $w v_0 v_0^* w^* = vv^*$, and $\|w - 1\| \leq \|vv^* - v_0 v_0^*\| < 2\epsilon$. Let $s = v v_0^* + (1 - vv^*) w$. It is apparent that this element is also a unitary of $\mathcal{A}$, which verifies

$$s v_0 = v.$$

Note that $s - 1 = v(v^* - v_0^*) + (1 - vv^*)(1 - w)$. Suppose that $\mathcal{A}$ is faithfully represented in H. Since v and $1 - vv^*$ have orthogonal ranges, if $\eta \in H$,

$$\|(s - 1)\eta\|^2 = \|v(v^* - v_0^*)\eta\|^2 + \|(1 - vv^*)(1 - w)\eta\|^2.$$

It follows that
$$\|s-1\| < \sqrt{3}\epsilon = 2\sin(r/4).$$
Then there exists a unique $a \in B_r(0)$, with $a = a_{\mathcal{H}} + a_{\mathcal{V}}$ as above, such that $s = exp(a) = e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}$. Note that $a_{\mathcal{V}}v_0 = 0$. Therefore
$$v = sv_0 = e^{a_{\mathcal{H}}}e^{a_{\mathcal{V}}}v_0 = e^{a_{\mathcal{H}}}v_0,$$
with $\|a_{\mathcal{H}}\| \leq 2\|a\| < 2r$. Take $\xi = a_{\mathcal{H}}$. Such ξ is clearly unique.

□

The above result states that R of 3.1 satisfies $R \geq 0.0034$.

We may combine these facts in our main result

Theorem 3.5 *Let $r > 0$ such that exp is a diffeomorphism on $B_r(0)$ ($r \leq d$). If $v, v_0 \in \mathcal{I}$, with $\|v - v_0\| < r' = \min\{1/2, \frac{2}{3\sqrt{3}}\sin(r/4)\}$, then there exists a unique geodesic with velocity vector $\xi = (\xi_1, \xi_2)$ such that $\|\xi_1\| < r'$, $\|\xi_2\| < \pi$, which joins v and v_0.*

4 Partial unitaries

Let us call *partial unitary* a partial isometry w such that the initial and final spaces coincide. Equivalently, $w^*w = ww^* = q$, or w is a unitary element of the reduced algebra $q\mathcal{A}q$. Let
$$\mathcal{I}_\Delta = \{w \in \mathcal{I} : ww^* = w^*w\} = \cup_{q\in\mathcal{P}} U_{q\mathcal{A}q}.$$
In [1] it is shown that this set $\mathcal{I}_\Delta$, or rather, its connected components, are C^∞ submanifolds of $\mathcal{I}$ (and of $\mathcal{A}$). In this section we study the properties of this set.

We shall construct a natural principal bundle over $\mathcal{I}_\Delta$ (which is not a homogeneous space). Fix a projection p in $\mathcal{A}$. Let
$$\Delta = \Delta^p = \{(\alpha, \beta) \in U_{\mathcal{A}} \times U_{\mathcal{A}} : \alpha p \alpha^* = \beta p \beta^*\}.$$
First note that Δ is not a subgroup of $U_{\mathcal{A}} \times U_{\mathcal{A}}$. It carries a right action from the group $U_{\mathcal{A}} \cap \{p\}' \times U_{\mathcal{A}} \cap \{p\}'$. We shall be interested in fact in a subgroup of this group and its right action on Δ. Consider the map
$$\pi_p^\Delta : \Delta \to \mathcal{I}_\Delta, \quad \pi_p^\Delta(\alpha, \beta) = \alpha p \beta^*,$$
which is given by the restriction to Δ of the former map π_p of section 2. Note that this map is indeed well defined, i.e. takes values in $\mathcal{I}_\Delta$: if $w = \alpha p \beta^*$ with $(\alpha, \beta) \in \Delta$, then $ww^* = \alpha p \alpha^* = \beta p \beta^* = w^*w$. Clearly this map is C^∞. Let us examine the fibre of π_p^Δ over p. If $\alpha p \beta^* = p$ then elementary computations show that α and β commute with p and $\alpha p = \beta p$. Let us denote by
$$G_p = \{(g_1, g_2) \in \Delta : g_1 p = p g_2\}.$$
It is apparent that G_p is a (Banach-Lie) subgroup of $U_{\mathcal{A}} \cap \{p\}' \times U_{\mathcal{A}} \cap \{p\}'$. Therefore it acts on Δ by right multiplication. Also, it is apparent that this action is free.

The range of π_p^Δ consists of all partial unitaries w such that their unit $ww^* = w^*w$ is unitarily equivalent to p. This set fills connected components [1].

Theorem 4.1 *The set Δ is a C^∞ submanifold of $\mathcal{A} \times \mathcal{A}$, and the map*
$$\pi_p^\Delta : \Delta \to \mathcal{I}_\Delta, \quad \pi_p^\Delta(\alpha, \beta) = \alpha p \beta^*$$
is a C^∞ principal bundle with structure group G_p.

Proof. The map $\pi : U_{\mathcal{A}} \to \mathcal{P}$, $\pi(u) = upu^*$ is a C^* submersion [13]. Therefore the map $\pi\times\pi : U_{\mathcal{A}} \times U_{\mathcal{A}} \to \mathcal{P}\times\mathcal{P}$, $(u, v) \mapsto (upu^*, vpv^*)$ is also a submersion. The subset $D = \{(q, q) : q \in \mathcal{P}\} \subset \mathcal{P}\times\mathcal{P}$ is clearly a submanifold of $\mathcal{P}\times\mathcal{P}$. It follows that $\Delta = (\pi\times\pi)^{-1}(D)$ is a submanifold of $U_{\mathcal{A}} \times U_{\mathcal{A}}$.

Let us exhibit trivializations for the map π_p^Δ. First we construct local cross sections. Fix $w_0 = \alpha_0 p\beta_0^*$. Let $w \in \mathcal{I}_\Delta$ such that $\|w - w_0\| < 1$. Then there exists a unitary $u = u(w_0, w) \in U_{\mathcal{A}}$, which is a C^∞ function on the parameter w such that $u\alpha_0 w_0^* w_0 \alpha_0^* u^* = p$ [1]. Let $w' = u\alpha_0 w\alpha_0^* u^*$. Clearly w' is a unitary of $p\mathcal{A}p$. Put

$$\alpha' = w'^2 + 1 - p, \quad \beta' = w' + 1 - p.$$

Then α', β' are unitaries in $\mathcal{A}$ which verify that $\alpha' p\beta'^* = w'$. Finally, put

$$\alpha = \alpha_0^* u^* \alpha', \quad \beta = \alpha_0^* u^* \beta'.$$

A straightforward computation shows that $\alpha p\beta^* = w$. Also note that

$$\alpha p\alpha^* = \alpha_0^* u^* \alpha' p\alpha'^* u\alpha_0 = \alpha_0 u^* pu\alpha_0 = w_0^* w_0.$$

Analogously $\beta p\beta^* = w_0^* w_0$. It follows that $(\alpha, \beta) \in \Delta$. In other words, we have found a C^∞ cross section $\sigma_{w_0}(w) = (\alpha, \beta) = (\alpha_w, \beta_w)$ for π_p^Δ, defined on the neighbourhood $\mathcal{U}_{w_0} = \{w \in \mathcal{I}_\Delta : \|w - w_0\| < 1\}$ of w_0 in $\mathcal{I}_\Delta$. Consider now the map

$$\phi_{w_0} : \mathcal{U}_{w_0} \times G_p \to (\pi_p^\Delta)^{-1}(\mathcal{U}_{w_0}),\ \phi_{w_0}(w, (g_1, g_2)) = \sigma_{w_0}(w)\cdot(g_1, g_2) = (\alpha_w g_1, \beta_w g_2).$$

Clearly it is well defined and smooth. Also the diagram

$$\begin{array}{ccc} \mathcal{U}_{w_0} \times G_p & \xrightarrow{\phi_{w_0}} & (\pi_p^\Delta)^{-1}(\mathcal{U}_{w_0}) \\ P_1 & \searrow & \Big\downarrow \pi_p^\Delta \\ & & \mathcal{U}_{w_0} \end{array}$$

commutes. Moreover, ϕ_{w_0} is a diffeomorphism. Its inverse is the map

$$\psi_{w_0} : (\pi_p^\Delta)^{-1}(\mathcal{U}_{w_0}) \to \mathcal{U}_{w_0} \times G_p, \quad \psi_{w_0}(\alpha, \beta) = \big(\alpha p\beta^*, (\hat{\alpha}^*\alpha, \hat{\beta}^*\beta)\big),$$

where $(\hat{\alpha}, \hat{\beta}) = \sigma_{w_0}(\alpha p\beta^*)$. Finally, it must be proven that the trivialization is equivariant with respect to the action of G_p. Indeed, if $(\alpha, \beta) \in \Delta$ and $(g_1, g_2) \in G_p$,

$$\psi_{w_0}((\alpha, \beta)\cdot(g_1, g_2)) = (\alpha g_1 p g_2^* \beta^*, (\widehat{\alpha g_1}^* \alpha g_1, \widehat{\beta g_2}^* \beta g_2))$$

$$= (\alpha p\beta^*, (\hat{\alpha}^*\alpha g_1, \hat{\beta}^*\beta g_2)) = \psi(\alpha, \beta)\cdot(g_1, g_2),$$

which holds because $(\widehat{\alpha g_1}, \widehat{\beta g_2}) = \sigma_{w_0}(\alpha g_1 p g_2^* \beta^*) = (\hat{\alpha}, \hat{\beta})$.

□

Next we shall introduce a connection in this principal bundle. To do so first we must compute the tangent spaces of Δ and the fibres $(\pi_p^\Delta)^{-1}(\{w\})$. Let $(\alpha(t), \beta(t))$ be a smooth curve in Δ with $\alpha(0) = \alpha$, $\beta(0) = \beta$, $\dot{\alpha}(0) = a$, $\dot{\beta}(0) = b$. First note that since $\alpha(t), \beta(t)$ are unitaries, then $\alpha^* a, a\alpha^*, \beta^* b, b\beta^*$ belong to $\mathcal{A}_{ah}$. Next, $\alpha(t)p\alpha^*(t) = \beta(t)p\beta^*(t)$ at $t = 0$ implies that $ap\alpha^* + \alpha pa^* = bp\beta^* + \beta pb^*$. Note that

$$ap\alpha^* + \alpha pa^* = (a\alpha^*)\alpha p\alpha^* + \alpha p\alpha^*(\alpha a^*) = (a\alpha^*)\alpha p\alpha^* + \alpha p\alpha^*(-a\alpha^*) = [a\alpha^*, \alpha p\alpha^*].$$

Analogously $bp\beta^* + \beta pb^* = [b\beta^*, \beta p\beta^*]$. Since $\alpha p\alpha^* = \beta p\beta^*$, one has

$$(T\Delta)_{(\alpha,\beta)} = \{(a,b) \in \mathcal{A} \times \mathcal{A} : a\alpha^*, b\beta^* \in \mathcal{A}_{ah}, [a\alpha^* - b\beta^*, \alpha p\alpha^*] = 0\}.$$

The tangent space of the fibre of π_p^Δ over $\alpha p\beta^*$ can be computed in a similar way. Let $(\omega_1(t), \omega_2(t))$ be a curve in such that $\omega_1(t)p\omega_2^*(t) = \alpha p\beta^*$, with $\dot{\omega_1}(0) = x, \dot{\omega_2}(0) = y$. This implies that $(\alpha^*\omega_1(t), \beta^*\omega_2(t)) \in G_p$. Therefore, denoting by $\Omega_{(\alpha,\beta)}$ the "vertical" space over $\alpha p\beta^*$, that is, the tangent space at (α, β) of the fibre of π_p^Δ, one gets

$$\Omega_{(\alpha,\beta)} = \{(x,y) \in \mathcal{A} \times \mathcal{A} : \alpha^*x, \beta^*y \in \mathcal{A}_{ah} \cap \{p\}', \text{ and } \alpha^*xp = \beta^*yp\}.$$

Equivalently, $\Omega_{(\alpha,\beta)} = \{(x,y) : (\alpha^*x, \beta^*y) \in \mathcal{G}_p\}$, where $\mathcal{G}_p$ denotes the Lie algebra of G_p.

A connection in the principal bundle consists of a distribution

$$(\alpha,\beta) \mapsto \mathcal{K}_{(\alpha,\beta)}$$

of subspaces of $(T\Delta)_{(\alpha,\beta)}$ with the following properties:

1. $\mathcal{K}_{(\alpha,\beta)} \oplus \Omega_{(\alpha,\beta)} = (T\Delta)_{(\alpha,\beta)}$.
2. The distribution is smooth, i.e. if (A, B) is a smooth tangent vector field on a neighbourhood of (α, β) in Δ, then the vector $(A_\mathcal{K}, B_\mathcal{K})$, which is the (pointwise) projection of (A, B) onto $\mathcal{K}$, is also smooth.
3. The distribution is equivariant under the action of G_p. Note that because the right action of G_p on Δ is in fact the restriction of a linear action, this property is equivalent to $\mathcal{K}_{(\alpha g_1,\beta g_2)} = \mathcal{K}_{(\alpha,\beta)} \cdot (g_1, g_2)$ for any $(g_1, g_2) \in G_p$.

An element α^*x commutes with p if and only if $x\alpha^*$ commutes with $\alpha p\alpha^*$, and analogously for β^*y. This makes apparent the fact that $\Omega_{(\alpha,\beta)} \subset (T\Delta)_{(\alpha,\beta)}$, moreover

$$\Omega_{(\alpha,\beta)} \subset \mathcal{Z} := \{(z_1, z_2) : z_1\alpha^*, z_2\beta^* \in \mathcal{A}_{ah} \text{ and commute with } \alpha p\alpha^*\} \subset (T\Delta)_{(\alpha,\beta)}.$$

In order to find a supplement for $\Omega_{(\alpha,\beta)}$ in $(T\Delta)_{(\alpha,\beta)}$ we shall supplement Ω in $\mathcal{Z}$ and add it to a supplement of $\mathcal{Z}$ in $(T\Delta)_{(\alpha,\beta)}$.

Observe that $(z_1, z_2) \in \mathcal{Z}$ if and only if α^*z_1 and β^*z_2 are antihermitic elements of $\mathcal{A}$ which commute with p. Therefore a natural supplement for $\Omega_{(\alpha,\beta)}$ in $\mathcal{Z}$ consists of all pairs of matrices (in terms of p) of the form

$$\alpha \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix} , \; \beta \begin{pmatrix} -c_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

Next we find a supplement for $\mathcal{Z}$ inside $(T\Delta)_{(\alpha,\beta)}$. Equivalently, a supplement for $\mathcal{Z} \cdot (\alpha^*, \beta^*)$ inside $(T\Delta)_{(\alpha,\beta)} \cdot (\alpha^*, \beta^*)$. These subspaces of $\mathcal{A}_{ah} \times \mathcal{A}_{ah}$ are, respectively,

$$\{(v_1, v_2) : [v_i, \alpha p\alpha^*] = 0, i = 1, 2\} \text{ and } \{(x_1, x_2) : [x_1 - x_2, \alpha p\alpha^*] = 0\}.$$

The condition $[x_1 - x_2, \alpha p\alpha^*] = 0$ means that in the decomposition (in terms of $\alpha p\alpha^*$) as diagonal and codiagonal matrices, the elements x_1 and x_2 have the same codiagonal part. A natural supplement for $\mathcal{Z}$ in $(T\Delta)_{(\alpha,\beta)}$ is the set

$$\{(a, a) : a \in \mathcal{A}_{ah} \text{ is codiagonal with respect to } \alpha p\alpha^*\} \cdot (\alpha, \beta).$$

Instead of describing explicitely the suplement $\mathcal{K}_{(\alpha,\beta)}$, let us write down the projection $P_{(\alpha,\beta)} = P_{\mathcal{K}_{(\alpha,\beta)}}$ that corresponds to this decomposition of $(T\Delta)_{(\alpha,\beta)}$. After routine calculations one gets

$$P_{(\alpha,\beta)}(x, y) = (\tilde{x}, \tilde{y}),$$

$$\tilde{x} = \frac{1}{2}(\alpha p\alpha^* x - \alpha p\beta^* y)p + \alpha p\alpha^* x(1-p) + (1-\alpha p\alpha^*)xp \tag{4.1}$$

$$\tilde{y} = \frac{1}{2}(\beta p\beta^* y - \beta p\alpha^* x)p + \beta p\beta^* y(1-p) + (1-\beta p\beta^*)yp. \tag{4.2}$$

It is apparent that the map $(\alpha,\beta) \mapsto P_{(\alpha,\beta)}$, as a map from Δ to $B_{\mathbb{R}}(\mathcal{A})$ is C^∞. Thus the smoothness requirement is fulfilled. Moreover, if $(g_1,g_2) \in G_p$ then $(\alpha g_1)p(\beta g_2)^* = \alpha p\beta^*$, $\beta g_2 p(\alpha g_1)^* = ((\alpha g_1 p(\beta g_2)^*)^* = (\alpha p\beta^*)^* = \beta p\alpha^*$, and $\alpha g_1 p(\alpha g_1)^* = \beta g_2 p(\beta g_2)^* = \alpha p\alpha^*$. Therefore, if $x,y \in (T\Delta)_{(\alpha,\beta)}$, and $P_{\alpha,\beta}(x,y) = (\tilde{x},\tilde{y})$ as above,

$$(\tilde{x},\tilde{y})\cdot(g_1,g_2) = P_{\alpha g_1,\beta g_2}(xg_1,yg_2),$$

(where $(xg_1,yg_2) \in (T\Delta)_{(\alpha g_1,\beta g_2)}$), i.e. the distribution $(\alpha,\beta)\mapsto\mathcal{K}_{(\alpha,\beta)}$ is G_p-invariant. Therefore we have the following:

Proposition 4.2 *Let $\mathcal{K}_{(\alpha,\beta)} = R(P_{(\alpha,\beta)})$. Then the distribution*

$$(\alpha,\beta)\mapsto\mathcal{K}_{(\alpha,\beta)}$$

defines a connection in the principal bundle π_p^Δ.

The differential of π_p^Δ at (α,β) is the map

$$\delta_{\alpha,\beta} : (T\Delta)_{(\alpha,\beta)} \to (T\mathcal{I}_\Delta)_w,\ \delta_{\alpha,\beta}(a,b) = ap\beta^* + \alpha pb^*,$$

where $w = \alpha p\beta^*$. Note that $\Omega_{(\alpha,\beta)}$ is the kernel of $\delta_{\alpha,\beta}$. It follows that

$$\delta_{\alpha,\beta}\big|_{\mathcal{K}_{\alpha,\beta}} : \mathcal{K}_{\alpha,\beta} \to (T\mathcal{I}_\Delta)_w$$

is a (real) linear isomorphism. A useful data of the connection is the distribution of inverses $\kappa_{\alpha,\beta} := \left(\delta_{\alpha,\beta}\big|_{\mathcal{K}_{\alpha,\beta}}\right)^{-1}$, which are given by

$$\kappa_{\alpha,\beta} : (T\mathcal{I}_\Delta)_w \to \mathcal{K}_{\alpha,\beta},\ \kappa_{\alpha,\beta}(z) = (\tilde{x},\tilde{y})$$

where

$$\tilde{x} = \frac{1}{2}\alpha p\alpha^* z\beta p + (1-\alpha p\alpha^*)z\beta p - \beta p\alpha^* z\alpha(1-p)$$

and

$$\tilde{y} = -\frac{1}{2}\beta p\alpha^* z\beta p - \beta p\alpha^* z\beta(1-p) + (1-\alpha p\alpha^*)z\alpha p.$$

Let us finish this section by computing the horizontal lifting differential equation of this connection. Fix $w = \alpha p\beta^*$, and $\gamma(t) \in \mathcal{I}_\Delta$, $t\in[0,1]$ a piecewise C^1 curve with $\gamma(0) = w$. We look for a piecewise C^1 curve $\Gamma(t) = (\Gamma_1(t),\Gamma_2(t)) \in \Delta$ such that Γ lifts γ and $\dot{\Gamma}$ is horizontal, i.e.

$$\pi_p^\Delta(\Gamma(t)) = \Gamma_1(t)p\Gamma_2^*(t) = \gamma(t)$$

and

$$\dot{\Gamma}(t) \in \mathcal{K}_{\Gamma(t)}, t\in[0,1].$$

Differentiating the first condition, one gets that $\delta_{\Gamma(t)}(\dot{\Gamma}(t)) = \dot{\gamma}(t)$, and since $\dot{\Gamma}(t)\in\mathcal{K}_{\Gamma(t)}$, applying $\kappa_{\Gamma(t)}$ one obtains the differential equation

$$\dot{\Gamma}(t) = \kappa_{\Gamma(t)}(\dot{\gamma}(t)), \tag{4.3}$$

or, explicitely, (omitting the parameter t)

$$\dot{\Gamma_1} = \frac{1}{2}\Gamma_1 p\Gamma_1^*\dot\gamma\Gamma_2 p + (1-\Gamma_1 p\Gamma_1^*)\dot\gamma\Gamma_2 p - \Gamma_2 p\Gamma_1^*\dot\gamma\Gamma_1(1-p),$$

$$\dot{\Gamma_2} = -\frac{1}{2}\Gamma_2 p\Gamma_1^*\dot\gamma\Gamma_2 p + (1-\Gamma_1 p\Gamma_1^*)\dot\gamma\Gamma_1 p - \Gamma_2 p\Gamma_1^*\dot\gamma\Gamma_2(1-p).$$

If we make the (a posteriori) assumption that Γ lifts γ, these equations may be rewritten, using that $\Gamma_1 p\Gamma_2^* = \gamma$ and $\Gamma_1 p\Gamma_1^* = \gamma^*\gamma = \gamma\gamma^* = \Gamma_2 p\Gamma_2^*$. Namely,

$$\dot{\Gamma_1} = \{\frac{1}{2}\gamma^*\gamma\dot\gamma\gamma^* + (1-\gamma^*\gamma)\dot\gamma\gamma^* - \gamma^*\dot\gamma(1-\gamma^*\gamma)\}\Gamma_1 \tag{4.4}$$

$$\dot{\Gamma_2} = \{-\frac{1}{2}\gamma^*\dot\gamma\gamma^*\gamma + (1-\gamma^*\gamma)\dot\gamma\gamma - \gamma^*\dot\gamma(1-\gamma^*\gamma)\}\Gamma_2. \tag{4.5}$$

We shall adopt these equations, and prove that the solutions lift γ horizontally. Note that these equations are linear, and therefore existence, and uniqueness under initial conditions, are granted.

We need the following result (see [31]): let $\dot\Omega = \Sigma\Omega$ be a linear differential equation in $\mathcal{A}$ such that $\Omega(t_0) \in U_{\mathcal{A}}$ and $\Sigma \in \mathcal{A}_{ah}$, then $\Omega(t) \in U_{\mathcal{A}}$ for all t.

To apply this result, note the following

Lemma 4.3 *Let γ be a piecewise C^1 curve in $\mathcal{I}_\Delta$. Then both*

$$\frac{1}{2}\gamma^*\gamma\dot\gamma\gamma^* + (1-\gamma^*\gamma)\dot\gamma\gamma^* - \gamma^*\dot\gamma(1-\gamma^*\gamma)$$

and

$$-\frac{1}{2}\gamma^*\dot\gamma\gamma^*\gamma + (1-\gamma^*\gamma)\dot\gamma\gamma^* - \gamma^*\dot\gamma(1-\gamma^*\gamma)$$

lie in $\mathcal{A}_{ah}$.

Proof. Note that both terms share the last two summands. To prove the lemma it suffices to show that

$$\frac{1}{2}\gamma^*\gamma\dot\gamma\gamma^*,\ -\frac{1}{2}\gamma^*\dot\gamma\gamma^*\gamma \text{ and } (1-\gamma^*\gamma)\dot\gamma\gamma^* - \gamma^*\dot\gamma(1-\gamma^*\gamma)$$

lie in $\mathcal{A}_{ah}$. First we deal with this last term. Differentiating $(1-\gamma^*\gamma)\gamma = 0$, we get

$$(1-\gamma^*\gamma)^{\cdot}\gamma + (1-\gamma^*\gamma)\dot\gamma = 0.$$

Multiplying by γ^* on the right (note that γ and γ^* commute),

$$(\gamma^*\gamma)^{\cdot}\gamma^*\gamma = -(1-\dot{\gamma^*\gamma})\gamma\gamma^* = (1-\gamma^*\gamma)\dot\gamma\gamma^*.$$

Analogously,

$$-\gamma^*\gamma(\gamma^*\gamma)^{\cdot} = -\gamma^*\dot\gamma(1-\gamma^*\gamma).$$

Then,

$$(1-\gamma^*\gamma)\dot\gamma\gamma^* - \gamma^*\dot\gamma(1-\gamma^*\gamma) = (\gamma^*\gamma)^{\cdot}\gamma^*\gamma - \gamma^*\gamma(\gamma^*\gamma)^{\cdot}$$

which is antihermitic, because $(\gamma^*\gamma)^{\cdot}$ is selfadjoint, being the derivative of a curve of selfadjoints. Next, note that $\gamma = \gamma\gamma^*\gamma$ implies that

$$\dot\gamma = \dot\gamma\gamma^*\gamma + \gamma\dot\gamma^*\gamma + \gamma\gamma^*\dot\gamma. \tag{4.6}$$

Multiplying this by γ^* on the right yields

$$\dot{\gamma}\gamma^* = \dot{\gamma}\gamma^* + \gamma\dot{\gamma}^*\gamma\gamma^* + \gamma\gamma^*\dot{\gamma}\gamma^*,$$

i.e.

$$\gamma\gamma^*\dot{\gamma}\gamma^* = -\gamma\dot{\gamma}^*\gamma\gamma^* = -(\gamma\gamma^*\dot{\gamma}\gamma^*)^*.$$

Analogously, one proves that $\gamma^*\dot{\gamma}\gamma^*\gamma \in \mathcal{A}_{ah}$. □

Theorem 4.4 *Let γ be a piecewise C^1 curve in $\mathcal{I}_\Delta$ with $\gamma(t_0) = w = \alpha p \beta$. Let Γ_1, Γ_2 be the unique solutions of, respectively*

$$\dot{\Gamma_1} = \{\frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\}\Gamma_1$$

and

$$\dot{\Gamma_2} = \{-\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)\}\Gamma_2,$$

with initial conditions $\Gamma_1(t_0) = \alpha$ and $\Gamma_2(t_0) = \beta$. Then $\Gamma = (\Gamma_1, \Gamma_2)$ is the horizontal lifting of γ with $\Gamma(t_0) = (\alpha, \beta)$.

Proof. Denote by

$$\Sigma_1 = \frac{1}{2}\gamma^*\gamma\dot{\gamma}\gamma^* + (1-\gamma^*\gamma)\dot{\gamma}\gamma^* - \gamma^*\dot{\gamma}(1-\gamma^*\gamma)$$

and

$$\Sigma_2 = -\frac{1}{2}\gamma^*\dot{\gamma}\gamma^*\gamma + (1-\gamma^*\gamma)\dot{\gamma}\gamma - \gamma^*\dot{\gamma}(1-\gamma^*\gamma).$$

The above lemmas prove that Γ_1, Γ_2 lie in $U_\mathcal{A}$. Let us prove that Γ lifts γ, i.e. $\Gamma_1 p \Gamma_2^* = \gamma$. Or, equivalently, $\Gamma_1^*\gamma\Gamma_2 = p$. If one differentiates the right hand side of this relation,

$$(\dot{\Gamma_1^*\gamma\Gamma_2}) = \dot{\Gamma_1^*}\gamma\Gamma_2 + \Gamma_1^*\dot{\gamma}\Gamma_1 + \Gamma_1^*\gamma\dot{\Gamma_2} = \Gamma_1^*(-\Sigma_1\gamma + \dot{\gamma} + \gamma\Sigma_2)\Gamma_2.$$

If we abreviate $q = \gamma^*\gamma = \gamma\gamma^*$, then

$$-\Sigma_1\gamma + \dot{\gamma} + \gamma\Sigma_2 = -\frac{1}{2}q\dot{\gamma}q - (1-q)\dot{\gamma}q + \dot{\gamma} - \frac{1}{2}q\dot{\gamma}q - q\dot{\gamma}(1-q) = (1-q)\dot{\gamma}(1-q).$$

Note that $\gamma(1-q) = \gamma(1-\gamma^*\gamma) = 0$. Also $(1-q)\gamma = 0$. Therefore

$$0 = \frac{d}{dt}\{(1-q)\gamma(1-q)\} = -\dot{q}\gamma(1-q) + (1-q)\dot{\gamma}(1-q) - (1-q)\gamma\dot{q} = (1-q)\dot{\gamma}(1-q).$$

Then $(\dot{\Gamma_1^*\gamma\Gamma_2}) = 0$, and $\Gamma_1^*(t_0)\gamma(t_0)\Gamma_2(t_0) = p$, imply $\Gamma_1 p \Gamma_2^* = \gamma$. Note that this implies in particular that $\Gamma \in \Delta$: $\Gamma_1 p \Gamma_1^* = \Gamma_1 p \Gamma_2^*(\Gamma_2 p \Gamma_1^*) = \gamma\gamma^* = \gamma^*\gamma = \Gamma_2 p \Gamma_2^*$.

Finally, we check that Γ is horizontal. Since we know that Γ lifts γ, we can undo the procedure which led to the equations 4.4 and 4.5, which are therefore equivalent to the condition

$$\dot{\Gamma} = \kappa_\Gamma(\dot{\gamma}) \in \mathcal{K}_\Gamma,$$

which says that Γ is horizontal. □

5 A linear connection in $\mathcal{I}_\Delta$

In this section we introduce a linear connection in $\mathcal{I}_\Delta$. We shall use the horizontal lifting equation in order to define a parallel transport in the tangent bundle of $\mathcal{I}_\Delta$. To do this one data is still missing: we need a way (a map) to translate elements from one horizontal space into another. Namely, we need a map $\mathcal{K}_{(\alpha,\beta)} \to \mathcal{K}_{(\delta,\epsilon)}$, for any $(\alpha,\beta),(\delta,\epsilon) \in \Delta$, which is equivariant under the action of G_p. We construct this map through an intermediate coordinate space. Consider the set

$$\mathcal{C} = \{(a,b) \in \mathcal{A}_{ah} \times \mathcal{A}_{ah} : (1-p)a(1-p) = (1-p)b(1-p) = 0, pap + pbp = 0 \text{ and } [a-b,p] = 0\}.$$

Lemma 5.1 *Let $(\alpha,\beta) \in \Delta$. Then*

$$c_{\alpha,\beta} : \mathcal{C} \to (\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)},$$

$$c_{\alpha,\beta}(a,b) = (a, pbp + \beta^*\alpha pb(1-p)\alpha^*\beta + \beta^*\alpha(1-p)bp\alpha^*\beta)$$

is an isomorphism, with inverse

$$c_{\alpha,\beta}^{-1}(x,y) = (x, pyp + \alpha^*\beta py(1-p)\beta^*\alpha + \alpha^*\beta(1-p)yp\beta^*\alpha).$$

Proof. That these maps above are one the inverse of the other is apparent, the only feature here being the fact that $\alpha^*\beta$ commutes with p. The fact that $c_{\alpha,\beta}$ maps $\mathcal{C}$ onto $(\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}$ needs a proof. Note from 4.1 and 4.2 that $(\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}$ consists of pairs $(\alpha^*\tilde{x},\beta^*\tilde{y})$ with

$$\tilde{x} = \frac{1}{2}(\alpha p\alpha^* x - \alpha p\beta^* y)p + \alpha p\alpha^* x(1-p) + (1-\alpha p\alpha^*)xp$$

and

$$\tilde{y} = \frac{1}{2}(\beta p\beta^* y - \beta p\alpha^* x)p + \beta p\beta^* y(1-p) + (1-\beta p\beta^*)yp.$$

Then clearly

$$\alpha^*\tilde{x} = \frac{1}{2}p(\alpha^* x)p - \frac{1}{2}p(\beta^* y)p + p(\alpha^* x)(1-p) + (1-p)(\alpha^* x)p,$$

where $\alpha^* x, \beta^* y \in \mathcal{A}_{ah}$, with an analogous expression for $\beta^*\tilde{y}$. With this description, note that $(\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}$ consists of pairs (r,s) of antihermitic matrices (in terms of p) satisfying the following relations:

1. $r_{11} + s_{11} = 0$
2. $r_{11} = s_{22} = 0$
3. $r - \alpha^*\beta s\beta^*\alpha$ commutes with p.

Consider $c_{\alpha,\beta}^{-1}(r,s) = (a,b)$. Clearly the two first relations are preserved. There is an alternate description for $c_{\alpha,\beta}$ (and for $c_{\alpha,\beta}^{-1}$), namely

$$c_{\alpha,\beta}^{-1}(r,s) = (r, \alpha^*\beta s\beta^*\alpha + psp - p\alpha^*\beta s\beta^*\alpha p).$$

Then $a - b = r - \alpha^*\beta s\beta^*\alpha + psp - p\alpha^*\beta s\beta^*\alpha p$, where $r - \alpha^*\beta s\beta^*\alpha$ commutes with p by the above description, and the other summands lie in $p\mathcal{A}p$. This proves that $c_{\alpha,\beta}^{-1}((\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}) \subset \mathcal{C}$. That $c_{\alpha,\beta}(\mathcal{C}) \subset (\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)}$ is proved analogously. □

Now we can introduce the transport map

$$T_{\alpha,\beta}^{\delta,\epsilon} : \mathcal{K}_{\alpha,\beta} \to \mathcal{K}_{\delta,\epsilon} \tag{5.7}$$

$$\mathcal{K}_{(\alpha,\beta)} \overset{l_{(\alpha^*,\beta^*)}}{\to} (\alpha^*,\beta^*)\cdot\mathcal{K}_{(\alpha,\beta)} \overset{c_{\alpha,\beta}}{\to} \mathcal{C} \overset{c^{-1}_{\delta,\epsilon}}{\to} (\delta^*,\epsilon^*)\cdot\mathcal{K}_{(\delta,\epsilon)} \overset{l_{(\delta,\epsilon)}}{\to} \mathcal{K}_{(\delta,\epsilon)}.$$

Explicitely,

$$T^{\delta,\epsilon}_{\alpha,\beta}(x,y) = (\delta\alpha^* x, \delta\alpha^* y\beta^*\alpha\delta^*\epsilon - \delta p\alpha^* y\beta^*\alpha\delta^*\epsilon p + \epsilon p\beta^* yp).$$

Note that this map has the following properties:

$$T^{\alpha,\beta}_{\alpha,\beta} = id \quad \text{and} \quad \left(T^{\delta,\epsilon}_{\alpha,\beta}\right)^{-1} = T^{\alpha,\beta}_{\delta,\epsilon}.$$

We must show that it is equivariant:

Proposition 5.2 *Let $(\alpha,\beta),(\delta,\epsilon)\in\Delta$ and $(g_1,g_2)\in G_p$. Then, if $v\in(T\mathcal{I}_\Delta)_w$ ($w=\alpha p\beta^*$),*

$$\left(T^{\delta,\epsilon}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))\right)\cdot(g_1,g_2) = T^{\delta g_1,\epsilon g_2}_{\alpha g_1,\beta g_2}(\kappa_{\alpha g_1,\beta g_2}(v)).$$

Proof. Let $(x,y)=\kappa_{\alpha,\beta}(v)$. Then $\kappa_{\alpha g_1,\beta g_2}(v)=(xg_1,yg_2)$ (Prop. 4.4). The proof proceeds by checking what happens if one replaces in the explicit version of T above, $(\alpha,\beta),(\delta,\epsilon)$ and (x,y) by (respectively) $(\alpha g_1,\beta g_2),(\delta g_1,\epsilon g_2)$ and (xg_1,yg_2). Apparently, one gets

$$T^{\delta g_1,\epsilon g_2}_{\alpha g_1,\beta g_2}(xg_1,yg_2) = \left(T^{\delta,\epsilon}_{\alpha,\beta}(x,y)\right)\cdot(g_1,g_2).$$

□

This property enables one to define the parallel transport of tangent vectors along piecewise smooth curves of $\mathcal{I}_\Delta$. Let $\gamma(t)$ be a piecewise C^1 curve of $\mathcal{I}_\Delta$, $t\in[0,1]$, with $\gamma(0)=w=\alpha p\beta^*$. Let $\Gamma(t)$ be the horizontal lifting of $\gamma(t)$, with $\Gamma(0)=(\alpha,\beta)$. Then we define

$$\tau_{\gamma(t)}:(T\mathcal{I}_\Delta)_w\to(T\mathcal{I}_\Delta)_{\gamma(t)},\ \ \tau_{\gamma(t)}(v)=\delta_{\Gamma(t)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))). \tag{5.8}$$

Theorem 5.3 *The map τ above is well defined (does not depend on the choice of (α,β) in the fibre of w), and is a linear isomorphism.*

Proof. It is apparent that τ is an isomorphism. Let us prove that it is well defined. Let $(\alpha g_1,\beta g_2)$ be another element in the fibre of w ($(g_1,g_2)\in G_p$). Then it is clear that $\Gamma(t)\cdot(g_1,g_2)$ is the solution of the horizontal lifting equations 4.4, 4.5 with initial conditions $\Gamma(0)\cdot(g_1,g_2)=(\alpha g_1,\beta g_2)$. If we compute $\tau_{\gamma(t)}$ using these data, we get

$$\delta_{\Gamma(t)\cdot(g_1,g_2)}(T^{\Gamma(t)\cdot(g_1,g_2)}_{\alpha g_1,\beta g_2}(\kappa_{\alpha g_1,\beta g_2}(v))),$$

which equals, by 5.2,

$$\delta_{\Gamma(t)\cdot(g_1,g_2)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))\cdot(g_1,g_2)).$$

Recall that $\delta_{(u_1,u_2)}(x_1,x_2)=x_1pu_2^*+u_1px_2^*$, and therefore $\delta_{(u_1g_1,u_2g_2)}(x_1g_1,x_2g_2)=x_1g_1pg_2^*u_2^*+u_1g_1pg_2^*x_2^*=\delta_{(u_1,u_2)}(x_1,x_2)$. Then the expression above equals

$$\delta_{\Gamma(t)}(T^{\Gamma(t)}_{\alpha,\beta}(\kappa_{\alpha,\beta}(v))).$$

□

The covariant derivative of a vector field $X=X_{\gamma(t)}$ tangent along a curve $\gamma(t)\in\mathcal{I}_\Delta$, $t\in[0,1]$, with $\gamma(0)=w=\alpha p\beta^*$ is given by:

$$\frac{DX}{dt}\Big|_{t=t_0} = \frac{d}{dt}\tau^{-1}_{\gamma(t)}(X_{\gamma(t)})\Big|_{t=t_0}.$$

If Γ is the horizontal lifting of γ with $\Gamma(0)=(\alpha,\beta)$, then $\tau^{-1}_\gamma(X_\gamma)=\delta_{(\alpha,\beta)}(T^{\alpha,\beta}_\Gamma(\kappa_\Gamma(X_\gamma)))$. In particular, we have the following

Proposition 5.4 *Let* $v \in (T\mathcal{I})_w$, *with* $w = \alpha p \beta^*$. *Then the unique geodesic* $\omega(t)$, $t \in \mathbb{R}$, *of this connection, with* $\omega(0) = w$ *and* $\dot{\omega}(0) = v$ *is given by*

$$\omega(t) = \Omega_1(t) p \Omega_2^*(t), \ t \in \mathbb{R},$$

where $\Omega = (\Omega_1, \Omega_2)$ *is characterized by*

$$\dot{\Omega}(t) = T_{\alpha,\beta}^{\Omega(t)}(\kappa_{\alpha,\beta}(v)), \ \Omega(0) = (\alpha, \beta),$$

where $\alpha p \beta^* = w$.

Proof. The geodesic ω satisfies $\frac{D\dot{\omega}}{dt} = 0$, i.e.

$$\frac{d}{dt}\big(\delta_{(\alpha,\beta)}(T_\Omega^{\alpha,\beta}(\kappa_\Omega(\dot{\omega})))\big) = 0,$$

Where Ω is the horizontal lifting of ω with initial condition $\Omega(0) = (\alpha, \beta)$. Recall 4.3 that $\kappa_\Omega(\dot{\omega}) = \dot{\Omega}$, then

$$0 = \frac{d}{dt}\big(\delta_{(\alpha,\beta)} \circ T_\Omega^{\alpha,\beta}(\dot{\Omega})\big) = \delta_{(\alpha,\beta)}\big(\frac{d}{dt}(T_\Omega^{\alpha,\beta}(\dot{\Omega})\big).$$

This derivative takes place in the Banach space $\mathcal{K}_{(\alpha,\beta)}$, on which $\delta_{(\alpha,\beta)}$ is an isomorphism. It follows that

$$\frac{d}{dt} T_\Omega^{\alpha,\beta}(\dot{\Omega}) = 0,$$

i.e. $T_\Omega^{\alpha,\beta}(\dot{\Omega})$ is constant and equals

$$T_{\Omega(0)}^{\alpha,\beta}(\dot{\Omega}(0)) = T_{\alpha,\beta}^{\alpha,\beta}(\kappa_{(\alpha,\beta)}(\dot{\omega}(0))) = \kappa_{(\alpha,\beta)}(v).$$

Then, using that $\big(T_\Omega^{\alpha,\beta}\big)^{-1} = T_{\alpha,\beta}^\Omega$, it follows t hat

$$\dot{\Omega} = T_{\alpha,\beta}^\Omega(\kappa_{(\alpha,\beta)}(v)).$$

$\square$

Remark 5.5 *In the proposition above it is stated that geodesics exist for all time* $t \in \mathbb{R}$. *This is clear if one makes the above equations explicit. Denote by* $(x_1, x_2) = \kappa_{\alpha,\beta}(v)$, *then* $\dot{\Omega} = T_{\alpha,\beta}^\Omega(x_1, x_2)$ *gives*

$$\dot{\Omega}_1 = \Omega_1 \alpha^* x_1 \ \textit{with} \ \Omega(0) = \alpha,$$

and

$$\dot{\Omega}_2 = \Omega_1 \alpha^* x_2 \beta^* \Omega_1^* \Omega_2 - \Omega_1 p \alpha^* x_2 \beta^* \alpha \Omega_1^* \Omega_2 p + \Omega_2 p \beta^* x_2 p \ \textit{with} \ \Omega_2(0) = \beta.$$

Note that $\Omega_1^*\Omega_2$ *commutes with* p, *because* $\Omega = (\Omega_1, \Omega_2) \in \Delta$. *Therefore the second summand on the right hand term can be written* $-\Omega_1 p \alpha^* x_2 \beta^* \alpha p \Omega_1^* \Omega_2$. *The first equation can be solved:*

$$\Omega_1(t) = \alpha e^{t\alpha^* x_1}.$$

This can be replaced in the second equation to obtain (with the modification pointed above)

$$\dot{\Omega}_2 = \alpha e^{t\alpha^* x_1} \alpha^* x_2 \beta^* \alpha e^{-t\alpha^* x_1} \alpha^* \Omega_2 - \alpha e^{t\alpha^* x_1} p \alpha^* x_2 \beta^* \alpha p e^{-t\alpha^* x_1} \alpha^* \Omega_2 + \Omega_2 p \beta^* x_2 p.$$

Using that $\alpha e^{t\alpha^* x_1} \alpha^* = e^{t x_1 \alpha^*}$, *the first two terms on the right hand side equal*

$$e^{t\alpha^* x_1} x_2 \beta^* e^{-t x_1 \alpha^*} \Omega_2 - e^{t\alpha^* x_1} \alpha p \alpha^* x_2 \beta^* \alpha p \alpha^* e^{-t x_1 \alpha^*} \Omega_2$$

$$= e^{t\alpha^* x_1}\{x_2\beta^* - \alpha p\alpha^* x_2\beta^*\alpha p\alpha^*\}e^{-tx_1\alpha^*}\Omega_2.$$

Since $(x_1, x_2) = \kappa_{\alpha,\beta}(v) \in \mathcal{K}_{(\alpha,\beta)}$*, it follows that (see 4.3)* $x_1\alpha^*$ *and* $x_2\beta^*$ *have the same codiagonal entries in their matrices in terms of* $\alpha p\alpha^*$*. Therefore*

$$x_2\beta^* - \alpha p\alpha^* x_2\beta^*\alpha p\alpha^* = x_1\alpha^* - \alpha p\alpha^* x_1\alpha^*\alpha p\alpha^*.$$

Then, the second equation is

$$\dot{\Omega}_2 = x_1\alpha^*\Omega_2 - e^{tx_1\alpha^*}\alpha p\alpha^* x_1 p\alpha^* e^{-tx_1\alpha^*}\Omega_2 + \Omega_2 p\beta^* x_2 p.$$

This is a linear differential equation, with solutions defined for all $t \in \mathbb{R}$*.*

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