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Integral Equations and Operator Theory

C*-Modular vector states

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Abstract. Let $\mathcal{B}$ be a C*-algebra and X a C* Hilbert $\mathcal{B}$-module. If $p \in \mathcal{B}$ is a projection, denote by $\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$, the p-sphere of X. For φ a state of $\mathcal{B}$ with support p in $\mathcal{B}$ and $x \in \mathcal{S}_p(X)$, consider the state φ_x of $\mathcal{L}_{\mathcal{B}}(X)$ given by $\varphi_x(t) = \varphi(\langle x, t(x)\rangle)$. In this paper we study the following sets associated to these states, and examine their topologic properties. For a fixed φ with support p,

$$\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}$$

and for all such φ,

$$\Sigma_{p,X} = \{\varphi_x : x \in \mathcal{S}_p(X) \text{ and } supp(\varphi) = p\}.$$

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1. Introduction

Let $\mathcal{B}$ be a C*- algebra, X a right C*-module over $\mathcal{B}$, and $\mathcal{L}_{\mathcal{B}}(X)$ the C*-algebra of adjointable operators of X. If $p \in \mathcal{B}$ is a projection, denote by $\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$ the p-sphere of X. We shall study the states of $\mathcal{L}_{\mathcal{B}}(X)$ which are *pure* in the modular sense. That is, for a state φ of $\mathcal{B}$ and a vector $x \in \mathcal{S}_p(X)$, we consider the state φ_x with density x, given by

$$\varphi_x(t) = \varphi(\langle x, t(x)\rangle) \quad t \in \mathcal{L}_{\mathcal{B}}(X).$$

We require that the state φ has support projection p.

If $x, y \in X$, let $\theta_{x,y} \in \mathcal{L}_{\mathcal{B}}(X)$ be the "rank one" operator given by $\theta_{x,y}(z) = x\langle y, z\rangle$. If $\langle x, x\rangle = p$ then the operator $\theta_{x,x} = e_x$ is a selfadjoint projection, and all projections arising in this manner, from vectors on $\mathcal{S}_p(X)$, are mutually (Murray-von Neumann) equivalent. It turns out that these modular, or *vector states* as we shall subsequently call them, are precisely the states of $\mathcal{L}_{\mathcal{B}}(X)$ with support of rank one, i.e. equal to one of these projections e_x.

We are interested in the following sets of states,

$$\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}$$

for φ a fixed state in $\mathcal{B}$, with support projection $\operatorname{supp}(\varphi) = p$ in $\mathcal{B}$,

$$\Sigma_p(\mathcal{B}) = \{\text{states of } \mathcal{B} \text{ with support } p\}$$

and

$$\Sigma_{p,X} = \{\psi_x : \psi \in \Sigma_p(\mathcal{B}), x \in \mathcal{S}_p(X)\}.$$

The (convex) set $\Sigma_p(\mathcal{B})$ is considered with the relative topology induced by the usual norm of the conjugate space of $\mathcal{B}$. The other two sets $\mathcal{O}_\varphi$ and $\Sigma_{p,X}$ are endowed with the following metrics.

1. - $\mathcal{O}_\varphi$ with the metric d_φ, $d_\varphi(\Phi, \Psi) = \inf\{\|x - y\| : \varphi_x = \Phi, \varphi_y = \Psi\}$.
2. - $\Sigma_{p,X}$ with the metric d, $d(\Phi, \Psi) = \|\Phi - \Psi\| + \|\operatorname{supp}(\Phi) - \operatorname{supp}(\Psi)\|$.

These metrics d and d_φ do come up naturally if one looks for continuity of the maps

$$\mathcal{S}_p(X) \to \mathcal{O}_\varphi \ , \ x \mapsto \varphi_x$$

and

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X} \ , \ (x, \varphi) \mapsto \varphi_x.$$

It turns out that these maps are also fibrations, a fact which enables one to examine the homotopy theory of these spaces, comparing it with the homotopy of the spheres $\mathcal{S}_p(X)$ studied in [1] and [2] For example, we obtain that if $p\mathcal{B}p$ is a von Neumann algebra and Xp is selfdual (as a $p\mathcal{B}p$-module), then $\mathcal{O}_\varphi$ is simply connected. Or that if $p\mathcal{B}p$ is a properly infinite von Neumann algebra, and again Xp is selfdual, then $\Sigma_{p,X}$ has trivial homotopy groups of all orders.

2. Preliminaries and notations

Let us establish some basic facts and notations about the vector states φ_x.

We shall be concerned with states of $\mathcal{B}$ that have their support in $\mathcal{B}$, a fact which holds automatically if $\mathcal{B}$ is a von Neumann algebra and φ is normal..

Each element $x \in \mathcal{S}_p(X)$ gives rise to a (non unital) $*$-isomorphism $i_x : p\mathcal{B}p \to \mathcal{L}_\mathcal{B}(X)$, $i_x(a) = \theta_{xa,x}$.

Fix $x_0 \in \mathcal{S}_p(X)$. Let us recall from [2] the principal fibre bundle, which we shall call the projective bundle

$$\rho : \mathcal{S}_p(X) \to \mathcal{E}_{e_{x_0}} = \{\text{projections in } \mathcal{L}_\mathcal{B}(X) \text{ equivalent to } e_{x_0}\}$$

given by $\rho(x) = e_x$. Note that $\mathcal{E}_{e_{x_0}}$ depends only on p and not on the choice of x_0 (all projections on $\mathcal{E}_{e_{x_0}}$ are of the form e_x for some $x \in \mathcal{S}_p(X)$). The structure group of the projective bundle is the unitary group $U_{p\mathcal{B}p}$ of $p\mathcal{B}p$.

As is usual notation, if φ is a faithful state of $\mathcal{B}$, $\mathcal{B}^\varphi$ is the centralizer algebra of φ, i.e. $\mathcal{B}^\varphi = \{a \in \mathcal{B} : \varphi(ab) = \varphi(ba) \text{ for all } b \in \mathcal{B}\}$. If the support $\operatorname{supp}(\varphi) = p < 1$, then denote by $\mathcal{B}_p^\varphi$ the centralizer of the restriction of φ to the reduced algebra $p\mathcal{B}p$.

Typically a, b, c will denote elements of $\mathcal{B}$, x, y, z elements of X and r, s, t elements of $\mathcal{L}_{\mathcal{B}}(X)$. $\mathcal{B}''$ will denote the von Neumann enveloping algebra of $\mathcal{B}$, and X' the selfdual completion of X, which is a C*-module over $\mathcal{B}''$ ([11]). By fibre bundle we mean a locally trivial fibre bundle, and by fibration we mean a surjective map having the homotopy lifting property ([13]).

Lemma 2.1. *Let φ be a state of $\mathcal{B}$ with supp$(\varphi) = p \in \mathcal{B}$, and x an element in $\mathcal{S}_p(X)$. Then supp$(\varphi_x) = e_x$.*

Proof. Clearly $\varphi_x(e_x) = \varphi(\langle x, e_x(x)\rangle) = \varphi(\langle x, xp\rangle) = \varphi(p) = 1$. Let us call $r = \text{supp}(\varphi_x) \in \mathcal{L}_{\mathcal{B}''}(X')$. We have $r \le e_x$, i.e. $e_x r = r e_x = r$. This implies that r is of the form $e_y = \theta_{y,y}$, namely, $y = r(x) \in X'$. Now $\langle r(x), r(x)\rangle = q$ is a projection in $\mathcal{B}''$, with $q \le p$. Indeed,

$$\begin{aligned}\langle r(x), r(x)\rangle\langle r(x), r(x)\rangle &= \langle r(x), r(x)\langle r(x), r(x)\rangle\rangle \\ &= \langle r(x), \theta_{r(x),r(x)}(r(x))\rangle = \langle r(x), r(x)\rangle.\end{aligned}$$

And $\langle r(x), r(x)\rangle p = \langle r(x), r(xp)\rangle = \langle r(x), r(x)\rangle$, i.e. $q \le p$. Now it is clear that $\varphi(q) = \varphi(\langle r(x), r(x)\rangle) = \varphi(\langle x, r(x)\rangle) = \varphi_x(r) = 1$, which implies that $q = p$, Therefore $\langle r(x) - x, r(x) - x\rangle = \langle r(x), r(x)\rangle + \langle x, x\rangle - \langle r(x), x\rangle - \langle x, r(x)\rangle = 0$, since all these products equal p (because $\langle r(x), x\rangle = \langle r^2(x), x\rangle = \langle r(x), r(x)\rangle$). Finally, $r(x) = x$ implies that $r = e_{r(x)} = e_x$. □

Lemma 2.2. *Let Φ be a state of $\mathcal{L}_{\mathcal{B}}(X)$ with supp$(\Phi) = e_x$ for some $x \in \mathcal{S}_p(X)$. Then $\Phi = \varphi_x$ for φ a state in $\mathcal{B}$ with supp$(\varphi) = p$. Namely $\varphi(a) = \Phi(i_x(a))$.*

Proof. Put $\varphi = \Phi \circ i_x$ as above. First note that if $t \in \mathcal{L}_{\mathcal{B}}(X)$, then $e_x t e_x = \theta_{x\langle x,t(x)\rangle,x}$. Then $\varphi_x(t) = \Phi(i_x(\langle x, t(x)\rangle)) = \Phi(\theta_{x\langle x,t(x)\rangle,x}) = \Phi(e_x t e_x) = \Phi(t)$. It remains to see that supp$(\varphi) = p$. Clearly $\varphi(p) = \Phi(\theta_{xp,x}) = \Phi(e_x) = 1$. Suppose that $q \le p$ is a projection in $\mathcal{B}''$ with $\varphi''(q) = \Phi(\theta_{xq,x}) = 1$ (φ'' here denotes the normal extension of the former φ to $\mathcal{B}''$). Note that $\theta_{xq,x} = \theta_{xq,xq} = e_{xq}$ is in fact a projection (associated to $xq \in \mathcal{S}_q(X')$), and verifies $e_{xq} \le e_x$. It follows that $\theta_{xq,xq} = \theta_{x,x}$. Then $xq = \theta_{xq,xq}(x) = \theta_{x,x}(x) = x$, and therefore $q = p$. □

Remark 2.3. If $\mathcal{B}$ is a von Neumann algebra, the inner product of X is weakly continuous and the state Φ of the preceding result is normal, then $\varphi = \Phi \circ i_x$ is also normal.

Proposition 2.4. *Let $\psi, \varphi \in \Sigma_p(\mathcal{B})$, $x, y \in \mathcal{S}_p(X)$. Then*

a) $\varphi_x = \psi_x$ *if and only if* $\varphi = \psi$.
b) $\varphi_x = \psi_y$ *if and only if* $\psi = \varphi \circ Ad(u)$, *with* $y = xu$ *and* $u \in U_{p\mathcal{B}p}$.
c) $\varphi_x = \varphi_y$ *if and only if* $y = xv$, *for* v *a unitary element in* $\mathcal{B}_p^{\varphi}$.

Proof. Let us start with a): $\varphi(b) = \varphi_x(\theta_{xb,x}) = \psi_x(\theta_{xb,x}) = \psi(b)$.

To prove b), suppose that $\varphi_x = \psi_y$. Then they have the same support, i.e. $e_x = e_y$, which implies that there exists a unitary element $u \in U_{p\mathcal{B}p}$ such that

$y = xu$ (see [2]). Then

$$\varphi_x(t) = \psi_y(t) = \psi(\langle xu, t(xu)\rangle) = \psi(u^*\langle x, t(x)\rangle u) = [\psi \circ Ad(u^*)]_x(t).$$

Using part a), this implies that $\varphi = \psi \circ Ad(u^*)$, or $\psi = \varphi \circ Ad(u)$.

To prove c), use b), and note that the unitary element $u \in U_{p\mathcal{B}p}$ satisfies $\varphi = \varphi \circ Ad(u)$, i.e. $u \in \mathcal{B}_p^\varphi$. □

3. The set $\mathcal{O}_\varphi$

In this section we shall consider the set $\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}$ for a fixed state φ of $\mathcal{B}$ with $\mathrm{supp}(\varphi) = p$. Note that in the particular case when $X = \mathcal{B}$ is a finite von Neumann algebra and $p = 1$, then $\mathcal{O}_\varphi$ is just the unitary orbit of φ. In the general case, there is a canonical map

$$\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi, \ \sigma(x) = \varphi_x.$$

Let us consider the following natural metric in $\mathcal{O}_\varphi$:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x' - y'\| : x', y' \in \mathcal{S}_p(X), \varphi_{x'} = \varphi_x, \varphi_{y'} = \varphi_y\}$$

It is clear that this metric induces the same topology as the quotient topology given by the map σ, also, that in view of 2.4 it can be computed as follows:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x - yv\| : v \text{ unitary in } \mathcal{B}_p^\varphi\}.$$

First note that this is indeed a metric. For instance, if $d_\varphi(\varphi_x, \varphi_y) = 0$ then there exist unitaries v_n in $\mathcal{B}_p^\varphi$ such that $\|x - yv_n\| \to 0$, i.e. $yv_n \to x$ in $\mathcal{S}_p(X)$. In particular yv_n is a Cauchy sequence, and therefore v_n is a Cauchy sequence, converging to a unitary v in $\mathcal{B}_p^\varphi$. Then $x = yv$ and $\varphi_x = \varphi_y$. The other properties follow similarly.

With this metric, $\mathcal{O}_\varphi$ is homeomorphic to the quotient $\mathcal{S}_p(X)/U_{\mathcal{B}_p^\varphi}$. The following result implies that the inclusion $\mathcal{O}_\varphi \subset \mathcal{B}^*$ (=conjugate space of $\mathcal{B}$) is continuous.

Lemma 3.1. *If $x, y \in \mathcal{S}_p(X)$, then $\|\varphi_x - \varphi_y\| \le 2\|x - y\|$. In particular*

$$\|\varphi_x - \varphi_y\| \le 2 d_\varphi(\varphi_x, \varphi_y)$$

where the norm $\| \ \|$ of the functionals denotes the usual norm of the conjugate space $\mathcal{B}^*$

Proof. If $t \in \mathcal{L}_\mathcal{B}(X)$, then $|\varphi_x(t) - \varphi_y(t)| \le |\varphi(\langle x, t(x-y)\rangle)| + |\varphi(\langle x-y, ty\rangle)|$. Now by the Cauchy-Schwarz inequality $\|\langle x, t(x-y)\rangle\| \le \|t\| \, \|x-y\|$, and $\|\langle x-y, ty\rangle\| \le \|x - y\| \, \|t\|$. Then $\|\varphi_x(t) - \varphi_y(t)\| \le 2\|t\| \, \|x - y\|$, and the result follows. □

We want the map $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$ to be a locally trivial fibre bundle. In order to obtain it we make the following assumption:

Hypothesis 3.2. There exists a conditional expectation $E_\varphi : p\mathcal{B}p \to \mathcal{B}_p^\varphi$.

This is the case if for example $\mathcal{B}$ is a von Neumann algebra and φ is normal. For the remaining of this section, we suppose that 3.2 holds.

Theorem 3.3. *The map $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$, $\sigma(x) = \varphi_x$ is a locally trivial fibre bundle. The fibre of this bundle is the unitary group $U_{\mathcal{B}_p^\varphi}$ of $\mathcal{B}_p^\varphi$.*

Proof. Since the spaces are homogeneous spaces, it suffices to show that there exist continuous local cross sections at every point x_0 of $\mathcal{S}_p(X)$. Suppose that $d_\varphi(\varphi_x, \varphi_{x_0}) < r < 1$. Then there exists a unitary operator v in $\mathcal{B}_p^\varphi$ such that $\|xv - x_0\| < 1$. Then

$$\|p - \langle xv, x_0\rangle\| = \|\langle x_0, x_0\rangle - \langle xv, x_0\rangle\| = \|\langle x_0 - xv, x_0\rangle\| \leq \|x_0 - xv\| < 1$$

It follows that $\langle xv, x_0\rangle$ is invertible in $p\mathcal{B}p$. Therefore, one can find r such that also $E_\varphi(\langle xv, x_0\rangle) = v^* E_\varphi(\langle x, x_0\rangle)$ is invertible in $\mathcal{B}_p^\varphi$. Then $E_\varphi(\langle x, x_0\rangle)$ is invertible. Let us put

$$\eta_{x_0}(\varphi_x) = x\mu(E_\varphi(\langle x, x_0\rangle))$$

defined on $\{\varphi_x : d_\varphi(\varphi_x, \varphi_{x_0}) < r\}$, where μ denotes the unitary part in the polar decomposition (of invertible elements) in $\mathcal{B}_p^\varphi$: $c = \mu(c)(c^*c)^{1/2}$. First note that η_{x_0} is well defined. If x' is a vector in the fibre of φ_x, then $x' = xv$ for $v \in U_{\mathcal{B}_p^\varphi}$. Then $x'\mu(E_\varphi(\langle x', x_0\rangle)) = xv\mu(v^* E_\varphi(\langle x, x_0\rangle)) = x\mu(E_\varphi(\langle x, x_0\rangle))$, where the last equality holds because $\mu(ua) = u\mu(a)$ if u is unitary. Next, $\eta_{x_0}(\varphi_{x_0}) = x_0\mu(E_\varphi(\langle x_0, x_0\rangle)) = x_0$, and η_{x_0} is a cross section for σ, because $\mu(E_\varphi(\langle x, x_0\rangle))$ is a unitary in $\mathcal{B}_p^\varphi$. Finally, let us see that η_{x_0} is continuous. Suppose that $d_\varphi(\varphi_{x_n}, \varphi_x) \to 0$, then there exist unitaries v_n in $\mathcal{B}_p^\varphi$ such that $x_n v_n \to x$. Then by the continuity of the operations, $x_n v_n \mu(E_\varphi(\langle x_n v_n, x_0\rangle)) = x_n\mu(E_\varphi(\langle x_n, x_0\rangle)) \to x\mu(E_\varphi(\langle x, x_0\rangle))$, i.e., $\eta_{x_0}(x_n) \to \eta_{x_0}(x)$. It is clear from 2.4 that the fibre is $U_{\mathcal{B}_p^\varphi}$. Namely, $\sigma^{-1}(\varphi_x) = \{xv : v \in U_{\mathcal{B}_p^\varphi}\}$. Note that $xv_n \to xv$ in $\sigma^{-1}(\varphi_x) \subset \mathcal{S}_p(X)$ if and only if $v_n \to v$ in $U_{\mathcal{B}_p^\varphi}$. □

We shall use the following result, which is a straightforward fact from the theory of fibrations

Lemma 3.4. *Suppose that one has the following commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & X \\ & \searrow \pi_2 & \downarrow p \\ & & Y, \end{array}$$

where E, X, Y are topological spaces, π_1, π_2 are fibrations and p is continuous and surjective. Then p is also a fibration.

There is another natural bundle associated to $\mathcal{O}_\varphi$, which is the mapping

$$\mathcal{O}_\varphi \to \mathcal{E}_e, \ \varphi_x \mapsto e_x,$$

where e is any projection of the form e_{x_0} for some $x_0 \in \mathcal{S}_p(X)$. Since $e_x = \mathrm{supp}(\varphi_x)$, we shall call this map supp. In general, taking support of positive functionals does not define a continuous map. However it is continuous in this context, i.e. restricted to the set $\mathcal{O}_\varphi$ with the metric d_φ. Indeed, as seen before, convergence of $\varphi_{x_n} \to \varphi_x$ in this metric implies the existence of unitaries v_n of $\mathcal{B}_p^\varphi \subset p\mathcal{B}p$ such that $x_n v_n \to x$ in $\mathcal{S}_p(X)$. This implies that $e_{x_n v_n} = e_{x_n} \to e_x$. Moreover, one has

Theorem 3.5. *The map supp : $\mathcal{O}_\varphi \to \mathcal{E}_e$ is a fibration with fibre $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. One has the following commutative diagram of fibre bundles*

$$\begin{array}{ccc} \mathcal{S}_p(X) & \xrightarrow{\rho} & \mathcal{O}_\varphi \\ & \searrow \sigma & \Big\downarrow supp \\ & & \mathcal{E}_e. \end{array}$$

Proof. That the diagram commutes is apparent. Since ρ and σ are fibre bundles, it follows using 3.4 that supp is a fibration. Note that if $e_x = e_y$, then there exists $u \in U_{p\mathcal{B}p}$ such that $y = xu$. Then $\varphi_y = \varphi_{xu} = (\varphi \circ Ad(u^*))_x$, therefore $\mathrm{supp}^{-1}(e_x) = \{(\varphi \circ Ad(u^*))_x : u \in U_{p\mathcal{B}p}\}$. In 2.4 it was shown that $\varphi_x = \psi_x$ (where φ and ψ are states of $\mathcal{B}$ with support p) implies $\varphi = \psi$. So $\{\varphi \circ Ad(u) : u \in U_{p\mathcal{B}p}\}$ parametrizes the fibres of supp, and clearly this set is in one to one correspondence with $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. Now $(\varphi \circ Ad(u_n))_x \to (\varphi \circ Ad(u))_x$ in $\mathcal{O}_\varphi$ if and only if $\inf_{v \in U_{\mathcal{B}_p^\varphi}} \|xu_n - xuv\| = \inf_{v \in U_{\mathcal{B}_p^\varphi}} \|u_n - uv\|$, i.e. the class of u_n tends to the class of u in $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$ (with the quotient topology induced by the norm of $\mathcal{B}$). □

One can use the homotopy exact sequences of these bundles to relate the homotopy groups of $\mathcal{O}_\varphi$, $\mathcal{S}_p(X)$, $\mathcal{E}_e$, $U_{p\mathcal{B}p}$, $U_{\mathcal{B}_p^\varphi}$ and $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. Namely:

$$\ldots \pi_n(U_{\mathcal{B}_p^\varphi}, p) \to \pi_n(\mathcal{S}_p(X), x_0) \xrightarrow{\sigma_*} \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \to \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p) \to \ldots$$

where x_0 is a fixed element in $\mathcal{S}_p(X)$, and

$$\ldots \pi_n(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \xrightarrow{\mathrm{supp}_*} \pi_n(\mathcal{E}, e_{x_0}) \to \pi_{n-1}(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \ldots$$

with φ a fixed state in $\Sigma_p(\mathcal{B})$.

The first result uses simply the fact that σ is continuous and surjective.

Corollary 3.6. *If $p\mathcal{B}p$ is a finite von Neumann algebra, then $\mathcal{O}_\varphi$ is arcwise connected.*

Proof. If $p\mathcal{B}p$ is finite, it was shown in [2] that $\mathcal{S}_p(X)$ is connected. □

Corollary 3.7. *If $p\mathcal{B}p$ is a von Neumann algebra and the restriction of φ to $p\mathcal{B}p$ is normal, then*

$$\pi_1(\mathcal{O}_\varphi, \varphi_x) \cong \pi_1(\mathcal{E}_e, e_x).$$

If moreover Xp is selfdual, then $\pi_1(\mathcal{O}_\varphi, \varphi_x) = 0$.

Proof. The proof follows by applying the tail of the homotopy exact sequence of the bundle supp, recalling from [4] that the fibre $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$ is simply connected. In the selfdual case, it was proven in [2] that the connected components of $\mathcal{E}_e$ are simply connected. □

Corollary 3.8. *If $p\mathcal{B}p$ is a von Neumann algebra, φ restricted to $p\mathcal{B}p$ is normal and Xp is selfdual, then, for a fixed $x_0 \in \mathcal{S}_p(X)$, the inclusion map*

$$i : U_{\mathcal{B}_p^\varphi} \hookrightarrow \mathcal{S}_p(X), \ v \mapsto x_0 v$$

induces an epimorphism

$$i_* : \pi_1(U_{\mathcal{B}_p^\varphi}, p) \to \pi_1(\mathcal{S}_p(X), x_0).$$

Proof. This time use the homotopy exact sequence of σ, and the fact that in this case $\pi_1(\mathcal{O}_\varphi, \varphi_{x_0}) = 0$. □

In other words, this result says that regardless of the size of the selfdual module X, any closed continuous curve $x(t) \in \mathcal{S}_p(X)$ with $x(0) = x(1) = x_0$ is homotopic to a closed curve of the form $x_0(t) = x_0 v(t)$, with $v(t)$ a curve of unitaries in $\mathcal{B}_p^\varphi$ with $v(0) = v(1) = p$.

Corollary 3.9. *Suppose that X is selfdual. If either*

a) *$p\mathcal{B}p$ is a properly infinite von Neumann algebra,*
or

b) *$p\mathcal{B}p$ is a von Neumann algebra of type II_1 with $\mathcal{L}_\mathcal{B}(X)$ properly infinite,*

then for $n \geq 1$

$$\pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \cong \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p).$$

Proof. In both cases, a) and b), one has that $\mathcal{S}_p(X)$ is contractible (see [2]). Therefore the proof follows writing down the homotopy exact sequence of the fibre bundle σ. □

Situation b) occurs for example if $p\mathcal{B}p$ is a II_1 factor and Xp is not finitely generated over $p\mathcal{B}p$.

Finally let us state an analogous result for general C*-algebras $\mathcal{B}$ (under the hypothesis 3.2) for the module $X = H_\mathcal{B} = \mathcal{B} \otimes \ell^2$. Here we use the fact [2], that $\mathcal{S}_p(H_\mathcal{B})$ is contractible. The proof follows similarly.

Corollary 3.10. *If 3.2 holds, and $X = H_\mathcal{B}$, then for $n \geq 1$*

$$\pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \cong \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p).$$

These two results establish that in these cases, if $U_{\mathcal{B}_p^\varphi}$ is connected, then $\pi_1(\mathcal{O}_\varphi, \varphi_{x_0})$ is trivial. This is granted if $p\mathcal{B}p$ is a von Neumann algebra. However, note that $\pi_2(\mathcal{O}_\varphi, \varphi_{x_0})$ is not trivial. This is because $\mathcal{B}_p^\varphi$ is a finite von Neumann algebra and therefore $U_{\mathcal{B}_p^\varphi}$ has non trivial fundamental group (see [7], [12]).

4. Vector states in $\mathcal{L}_{\mathcal{B}}(X)$

In 2.2 it was shown that a state Φ of $\mathcal{L}_{\mathcal{B}}(X)$ with support $e = e_x$ for some $x \in \mathcal{S}_p(X)$ is of the form $\Phi = \varphi_x$ for some state φ in $\mathcal{B}$ with support p. Recall that we denote by $\Sigma_p(\mathcal{B})$ the set of states of $\mathcal{B}$ with support p, and by $\Sigma_{p,X}$ the set of states of $\mathcal{L}_{\mathcal{B}}(X)$ with support equivalent to e. In other words, $\Sigma_{p,X} = \cup_{\varphi \in \Sigma_p(\mathcal{B})} \mathcal{O}_\varphi$. One has the assignment

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X} \quad , \quad (x, \varphi) \mapsto \varphi_x.$$

Remember that $\varphi_x = \psi_y$, with $\varphi, \psi \in \Sigma_p(\mathcal{B})$, $x, y \in \mathcal{S}_p(X)$ if and only if $\psi = \varphi \circ Ad(u)$ with $y = xu$ and $u \in U_{p\mathcal{B}p}$ (see 2.4 part c)).

The unitary group $U_{p\mathcal{B}p}$ acts both on $\mathcal{S}_p(X)$ and $\Sigma_p(\mathcal{B})$. We may consider the diagonal action on $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$, defined by $u.(x, \varphi) = (xu, \varphi \circ Ad(u))$. It follows that if we denote the quotient

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) / \{(x, \varphi) \sim (xu, \varphi \circ Ad(u)), u \in U_{p\mathcal{B}p}\} := \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$$

(as is usual notation), then the assignment above induces a bijection

$$\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \simeq \Sigma_{p,X}.$$

If we endow $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$ with the quotient topology (where $\mathcal{S}_p(X)$ and $\Sigma_p(\mathcal{B})$ are considered with the norm topologies), a natural question is what topology does this bijection induce in $\Sigma_{p,X}$. The following result states that convergence of a sequence in the quotient topology is equivalent in $\Sigma_{p,X}$ to convergence (in norm) of the states and their supports.

Proposition 4.1. *Consider in $\Sigma_{p,X}$ the metric d given by*

$$d(\Phi, \Psi) = \|\Phi - \Psi\| + \|supp(\Phi) - supp(\Psi)\|.$$

Then the metric space $(\Sigma_{p,X}, d)$ is homeomorphic to $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$, where the homeomorphism is given by the above bijection.

Proof. Denote by $[(x, \varphi)]$ the class of (x, φ) in $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Suppose that $[(x_n, \varphi_n)]$ converge to $[(x, \varphi)]$ in $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Then there exist unitaries u_n in $p\mathcal{B}p$ such that $x_n u_n$ tends to x and $\varphi_n \circ Ad(u_n)$ tends to φ in the respective norms. By continuity of the inner product, it is clear then that $e_{x_n} = \theta_{x_n, x_n} = \theta_{x_n u_n, x_n u_n} \to e_x$ and $\varphi_{n x_n} = (\varphi_n \circ Ad(u_n))_{x_n u_n} \to \varphi_x$, and therefore the assignment $[(x, \varphi)] \mapsto \varphi_x$ is continuous. On the other direction, suppose that $d(\Phi_n, \Phi)$ tends to zero. There exist $\varphi_n, \varphi \in \Sigma_p(\mathcal{B})$ and $x_n, x \in \mathcal{S}_p(X)$ such that $\Phi_n = \varphi_{n x_n}$ and $\Phi = \varphi_x$. We have that $\mathrm{supp}(\Phi_n) = e_{x_n} \to \mathrm{supp}(\Phi) = e_x$. Now $e_{x_n} = \rho(x_n)$, $e_x = \rho(x)$, and $\rho : \mathcal{S}_p(X) \to \mathcal{E}_e$ is a fibre bundle with fibre $U_{p\mathcal{B}p}$, therefore there exist unitaries u_n in $p\mathcal{B}p$ such that $x_n u_n \to x$. We may replace the x_n by $y_n = x_n u_n$ and φ_n by $\psi_n = \varphi_n \circ Ad(u_n)$, and still have $\Phi_n = \psi_{n y_n}$, with $y_n \to x$. We claim that $\psi_n \to \varphi$. Indeed, if $a \in \mathcal{B}$, by a typical argument

$$|\psi_n(a) - \varphi(a)| = |\Phi_n(\theta_{y_n a, y_n}) - \Phi(\theta_{xa,x})| \leq \|\Phi_n\| \|\theta_{y_n a, y_n} - \theta_{xa,x}\| + \|\Phi_n - \Phi\| \|\theta_{xa,x}\|.$$

The first summand is bounded by (using the Cauchy-Schwarz inequality)

$$\|\theta_{y_n a, y_n} - \theta_{xa,x}\| \leq \|\theta_{y_n a, y_n - x}\| + \|\theta_{y_n a - xa, x}\| \leq \|y_n\|\|a\|\|y_n - x\| + \|y_n - x\|\|a\|\|x\|,$$

which equals $2\|a\|\|y_n - x\|$. The other summand equals $\|\Phi_n - \Phi\|\|a\|$. It follows that $[(y_n, \psi_n)] = [(x_n, \varphi_n)] \to [(x, \varphi)]$. □

Since $\|\Phi - \Psi\| \leq d(\Phi, \Psi)$, it is clear that the inclusion of $(\Sigma_{p,X}, d)$ in $(\mathcal{L}_{\mathcal{B}}(X)^*, \|\ \|)$ is continuous. The following example shows that the topology given by the metric d in $\Sigma_{p,X}$ does not coincide with the norm topology of the conjugate space of $\mathcal{L}_{\mathcal{B}}(X)$. In other words, that convergence of the vector states (which a priori have equivalent supports) does not imply convergence of the supports.

Example. Let $\mathcal{B} = D \subset B(\ell^2(\mathbb{N}))$ be the subalgebra of diagonal matrices (with respect to the canonical basis). Consider the conditional expectation $E : B(\ell^2(\mathbb{N})) \to D$ which consists on taking the diagonal entries. Let $a \in D$ be a trace class positive diagonal operator with trace one, and no zero entries in the diagonal. Put $\varphi(x) = Tr(ax)$, $x \in B(\ell^2(\mathbb{N}))$. Clearly, φ is faithful and $B(\ell^2(\mathbb{N}))^{\varphi} = D$. Let b be the unilateral shift in $\ell^2(\mathbb{N})$. Denote by q_n the $n \times n$ Jordan nilpotent, and w_n the unitary operator on $\ell^2(\mathbb{N})$ having the unitary matrix $q_n + {q_n^*}^{n-1}$ on the first $n \times n$ corner and the rest of the diagonal completed with 1.

We shall consider the D-right module X as the completion of $B(\ell^2(\mathbb{N}))$ with the D-valued inner product given by E, i.e. $\langle x, y \rangle = E(x^*y)$, $x, y \in B(\ell^2(\mathbb{N}))$. Note that X is also a $B(\ell^2(\mathbb{N}))$-left module, and so the elements of $B(\ell^2(\mathbb{N}))$ act as adjointable operators in X. Consider the faithful state φ of D equal to the restriction of the former φ. The elements w_n and b lie in the unit sphere of X.

We claim that the projections e_{w_n} do not converge to e_b. Suppose that they do converge, again using that the map $x \mapsto e_x$ is a bundle, there would exist unitaries $v_n \in D$ such that $w_n v_n \to b$ in $\mathcal{S}_1(X)$. In [4] it was shown that the element b cannot be approximated by unitaries of $B(\ell^2(\mathbb{N}))$ in the norm topology of the module X.

On the other hand, the states φ_{w_n} converge to φ_b in the norm topology of the conjugate space of $\mathcal{L}_{\mathcal{B}}(X)$. Indeed

$$|\varphi_{w_n}(t) - \varphi_b(t)| = |Tr\,(a(\langle w_n, t(w_n)\rangle - \langle b, t(b)\rangle))|$$

$$\leq |Tr\,(a(\langle w_n, t(w_n) - t(b)\rangle))| + |Tr\,(a(\langle w_n - b, t(b)\rangle))|\,.$$

The first summand can be bounded by $\|t\|\ Tr(a(2 - E(w_n^* b) - E(b^* w_n)))$. Since $Tr(a) = 1$ and E is trace invariant, this term equals

$$\|t\|\ \left(2 - Tr(a w_n^* b + a b^* w_n)\right) = 2\|t\| \sum_{k \geq n} a_k,$$

where a_k are the diagonal entries of a. It is clear that this term tends to zero when $n \to \infty$. The other summand can be dealt in a similar way, establishing our claim.

Summarizing, the states φ_{w_n} converge but their supports e_{w_n} do not.

Next we shall see that the quotient map

$$\wp_1 : \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \quad , \quad \wp_1(x,\varphi) = [(x,\varphi)]$$

and the projection

$$\wp_2 : \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \to \mathcal{S}_p(X)/U_{p\mathcal{B}p} \quad , \quad \wp_2([(x,\varphi)]) = [x]$$

are fibrations. Equivalently, if $\Sigma_{p,X}$ is considered with the topology induced by the metric d, the maps $(x,\varphi) \mapsto \varphi_x$ and $\varphi_x \mapsto e_x$ are fibrations. In what follows, for brevity, we shall use $\Sigma_{p,X}$ (considered with the metric d) instead of $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Therefore $\wp_1(x,\varphi) = \varphi_x$.

Theorem 4.2. *The map $\wp_1 : \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X}$, $\wp_1(x,\varphi) = \varphi_x$ is a principal fibre bundle with fibre $U_{p\mathcal{B}p}$.*

Proof. It suffices to exhibit a local cross section around a generic base point φ_x. We claim that there is a neighborhood of φ_x such that elements ψ_y in this neighborhood satisfy that $\langle y,x\rangle$ is invertible. Indeed, if $d(\varphi_x,\psi_y) < r$, then $\|e_x - e_y\| < r$. If we choose r small enough so that e_y lies in the ball around e_x in which a local cross section of $\rho(x) = e_x$ is defined, then there exists a unitary u in $p\mathcal{B}p$ such that $\|x - yu\| < 1$. Note that

$$\|p - \langle yu, x\rangle\| = \|\langle x - yu, x\rangle\| \le \|x - yu\| < 1.$$

Then $\langle yu, x\rangle = u^*\langle y,x\rangle$ is invertible in $p\mathcal{B}p$, and therefore also $\langle y,x\rangle$. In this neighborhood put

$$s(\psi_y) = (y\mu(\langle y,x\rangle), \psi \circ Ad(\mu(\langle y,x\rangle))),$$

where μ denotes the unitary part in the polar decomposition of invertible elements in $p\mathcal{B}p$ as before. We claim that s is well defined, is a local cross section and is continuous.

Suppose that $\psi_y = \psi'_{y'}$, then there exits a unitary u in $p\mathcal{B}p$ such that $y' = yu$ and $\psi = \psi' \circ Ad(u)$. Then $y'\mu(\langle y',x\rangle) = yu\mu(u^*\langle y,x\rangle) = y\mu(\langle y,x\rangle)$. Also, $\psi' \circ Ad(\mu(\langle y',x\rangle)) = \psi \circ Ad(u) \circ Ad(\mu(u^*\langle y,x\rangle)) = \psi \circ Ad(u) \circ Ad(u^*) \circ Ad(\mu(\langle y,x\rangle)) = \psi \circ Ad(\mu(\langle y,x\rangle))$.

That it is a cross section is apparent. Let us see that s is continuous. Suppose that $\psi_{n y_n} \to \varphi'_{x'}$ for $\varphi'_{x'}$ in the neighborhood of φ_x where s is defined. This implies that there exist unitaries u_n in $U_{p\mathcal{B}p}$ such that $y_n u_n \to x'$ and $\psi_n \circ Ad(u_n) \to \varphi'$ in the norm topologies. The continuity of the inner product implies that $y_n u_n \mu(\langle y_n u_n, x\rangle) = y_n\mu(\langle y_n,x\rangle) \to x'\mu(\langle x',x\rangle)$. Also $\psi_n \circ Ad(u_n) \circ Ad(\mu(\langle y_n u_n, x\rangle)) = \psi_n \circ Ad(\mu(\langle y_n,x\rangle)) \to \varphi' \circ Ad(\mu(\langle x',x\rangle))$. □

Next we consider the map $\wp_2 : \Sigma_{p,X} \to \mathcal{S}_p(X)/U_{p\mathcal{B}p}$. Recall that $\mathcal{S}_p(X)/U_{p\mathcal{B}p}$ is homeomorphic to $\mathcal{E}_e$, where the homeomorphism is given by $[x] \mapsto e_x$. The following result states that taking support of a state in $\Sigma_{p,X}$ (regarded with the d topology) is a fibration.

Theorem 4.3. *The map $\wp_2 : \Sigma_{p,X} \to \mathcal{S}_p(X)/U_{p\mathcal{B}p}$, given by $\wp_2(\varphi_x) = [x]$ is a fibration with fibre $\Sigma_p(\mathcal{B})$.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) & \stackrel{\wp_1}{\longrightarrow} & \Sigma_{p,X} \\ & \searrow p & \Big\downarrow \wp_2 \\ & & \mathcal{S}_p(X)/U_{p\mathcal{B}p}, \end{array}$$

where p is given by $p(x,\varphi) = [x]$. Clearly p is a fibre bundle, because it is the composition of the projective bundle $x \mapsto [x]$ with the projection $(x,\varphi) \mapsto x$. The map $\wp_1$ was shown to be a fibration. It follows from 3.4 the $\wp_2$ is a fibration. The fibre $\wp_2^{-1}([x])$ consists of all states φ_y with $[y] = [x]$. Then there exists $u \in U_{p\mathcal{B}p}$ such that $\varphi_y = (\varphi \circ Ad(u^*))_x$, so that one may fix x (and not just $[x]$). Now $\varphi_x = \psi_x$ implies $\varphi = \psi$. It follows that the fibre over $[x]$ is the set $\{\varphi_x : \varphi \in \Sigma_p(\mathcal{B})\}$, which identifies with $\Sigma_p(\mathcal{B})$. □

We will use the fibrations $\wp_1$ and $\wp_2$ to obtain information about the homotopy type of these spaces.

As in the previous section, applying the homotopy exact sequences of these fibrations, one obtains

$$\begin{aligned} \ldots \pi_n\left(U_{p\mathcal{B}p}, p\right) \to \quad & \pi_n(\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}), (x_0, \varphi)) \stackrel{(\wp_1)_*}{\to} \\ & \stackrel{(\wp_1)_*}{\to} \pi_n(\Sigma_{p,X}, \varphi_{x_0}) \to \pi_{n-1}(U_{p\mathcal{B}p}, p) \to \ldots \end{aligned}$$

and

$$\ldots \pi_n(\Sigma_p(\mathcal{B}), \varphi) \to \pi_n(\Sigma_{p,X}, \varphi_{x_0}) \stackrel{(\wp_2)_*}{\to} \pi_n(\mathcal{E}, e_{x_0}) \to \pi_{n-1}(\Sigma_p(\mathcal{B}), \varphi) \ldots$$

First note that since $\Sigma_p(\mathcal{B})$ is convex, $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$ has the same homotopy type as $\mathcal{S}_p(X)$, and

$$\pi_*(\Sigma_{p,X}) = \pi_*(\mathcal{E}_{e_x}).$$

Corollary 4.4. *The space $\mathcal{S}_p(H_\mathcal{B}) \times \Sigma_p(\mathcal{B})$ is contractible.*

Proof. It was remarked in the preceding section that $\mathcal{S}_p(H_\mathcal{B})$ is contractible. □

Corollary 4.5. *For $\varphi_0 \in \Sigma_p(\mathcal{B})$ and $x_0 \in \mathcal{S}_p(H_\mathcal{B})$ fixed,*

$$\pi_n\left(\Sigma_{p,H_\mathcal{B}}, \varphi_{0_{x_0}}\right) \cong \pi_{n-1}(U_{p\mathcal{B}p}, p), \quad n \geq 1.$$

In particular, if $U_{p\mathcal{B}p}$ is connected, $\pi_1\left(\Sigma_{p,H_\mathcal{B}}, \varphi_{0_{x_0}}\right) = 0$. If moreover $p\mathcal{B}p$ is a properly infinite von Neumann algebra, $\Sigma_{p,H_\mathcal{B}}$ has trivial homotopy groups of all orders.

Proof. The first fact follows from the contractibility of $\mathcal{S}_p(H_\mathcal{B}) \times \Sigma_p(\mathcal{B})$, which implies that in the homotopy sequence $\pi_k(\mathcal{S}_p(H_\mathcal{B}) \times \Sigma_p(\mathcal{B}), (\varphi_0, x_0)) = 0$ for all k. The second fact follows using that $U_{p\mathcal{B}p}$ is connected. Using that ([6]) if $p\mathcal{B}p$ is properly infinite, then $U_{p\mathcal{B}p}$ is contractible, it follows that

$$\pi_n(\Sigma_{p,H_\mathcal{B}}, \varphi_{0_{x_0}}) = 0$$

for all $n \geq 0$. □

One can be more specific, since the homotopy groups of the unitary group of a C*-algebra (at least for $n = 1$) have been computed in many cases ([7], [12], [15]). For example, in the von Neumann algebra case, one can compute the fundamental group of the unitary group in terms of the type decomposition of the algebra.

Corollary 4.6. *If $p\mathcal{B}p$ is a finite von Neumann algebra, then $\Sigma_{p,X}$ is connected.*

Proof. It was noted before that if $p\mathcal{B}p$ is finite, then $\mathcal{S}_p(X)$ is connected. □

Corollary 4.7. *If $p\mathcal{B}p$ is a properly infinite algebra, then for $n \geq 0$*

$$\pi_n\left(\Sigma_{p,X}, \varphi_{0_{x_0}}\right) \cong \pi_n(\mathcal{S}_p(X), x_0).$$

If moreover Xp is selfdual, then

$$\pi_n\left(\Sigma_{p,X}, \varphi_{0_{x_0}}\right) = 0$$

for all $n \geq 0$.

Proof. The proof follows writing the homotopy exact sequence of $\wp_1$. If $p\mathcal{B}p$ is properly infinite, its unitary group is contractible. If moreover Xp is selfdual, it was pointed out before that $\mathcal{S}_p(X)$ is contractible. □

We turn now our attention to the bundle $\wp_2$.

Corollary 4.8. *If $p\mathcal{B}p$ is a von Neumann algebra and Xp is selfdual, then the group $\pi_1(\Sigma_{p,X}, \varphi_x)$ is trivial.*

Proof. It was shown in [2] that $\pi_1(\mathcal{E}_{e_x}) = 0$ □

Remark 4.9. There is another map related to this situation, namely the other projection $\wp_3$,

$$\wp_3 : \Sigma_{p,X} \to \Sigma_p(\mathcal{B})/U_{p\mathcal{B}p}, \ \ \wp_3(\varphi_x) = [\varphi].$$

This map is well defined and continuous, and if one goes back to the notation $\mathcal{S}_p(X) \times_{u_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$, $\wp_3$ is the map $(x, \varphi) \mapsto \varphi$ at the quotient level,

$$[(x, \varphi)] \mapsto [\varphi].$$

However this map is not, in general, even a weak fibration. To see this consider the case when $X = \mathcal{B}$ is a finite algebra, and $p = 1$. Here $\mathcal{L}_\mathcal{B}(\mathcal{B}) = \mathcal{B}$ and $\Sigma_{1,\mathcal{B}}$ consist of the states of $\mathcal{B}$ with support equivalent to 1 (note that $x \in \mathcal{S}_1(X)$ verifies $x^*x = 1$, i.e. $x \in U_\mathcal{B}$, and $e_x = 1$). That is, $\Sigma_{1,\mathcal{B}}$ is the set of faithful states of $\mathcal{B}$ ($= \Sigma_1(\mathcal{B})$ in our notation). It follows that $\wp_3$ is just the quotient map

$$\Sigma_1(\mathcal{B}) \to \Sigma_1(\mathcal{B})/U_\mathcal{B}.$$

Moreover, take $\mathcal{B} = M_n(\mathbb{C})$ ($n < \infty$). Then the quotient map above is not a weak fibration. Indeed, both sets $\Sigma_1(M_n(\mathbb{C}))$ and $\Sigma_1(M_n(\mathbb{C}))/U_{M_n(\mathbb{C})}$ are convex metric spaces. The latter can be identified, using the density matrices, to the n-tuples of eigenvalues $(\lambda_1, ..., \lambda_n)$ arranged in decreasing order and normalized such that $\sum \lambda_k = 1$, with the ℓ_1 distance. If this quotient map were a weak fibration, then the fibre would have trivial homotopy groups of all orders. This is clearly not the case, since the fibre is the unitary group $U(n)$ of $M_n(\mathbb{C})$.

Remark 4.10. The set $\mathcal{O}_\varphi$ lies inside $\Sigma_{p,X}$, namely as the states φ_x with φ fixed. If one regards $\mathcal{O}_\varphi$ with the metric d_φ and $\Sigma_{p,X}$ with the metric d, it is clear that the inclusion is continuous. Indeed, it was noted that supp is continuous in $\mathcal{O}_\varphi$. Therefore if $d_\varphi(\varphi_{x_n}, \varphi_x) \to 0$, then $e_{x_n} \to e_x$, which implies that $d(\varphi_{x_n}, \varphi_x) \to 0$.

However, the identity mapping $(\mathcal{O}_\varphi, d_\varphi) \to (\mathcal{O}_\varphi, d)$ is not (in general) a homeomorphism. Indeed, take $X = \mathcal{B}$ and φ faithful. Then $\mathcal{O}_\varphi$ is the unitary orbit $\{\varphi \circ Ad(u) : u \in U_\mathcal{B}\} \sim U_\mathcal{B}/U_{\mathcal{B}^\varphi}$, and it is clear that d_φ induces the same topology as the quotient topology ($U_\mathcal{B}$ with the norm topology). On the other hand, $\Sigma_{p,X}$ coincides in this case with $\Sigma_1(\mathcal{B})$ the set of faithful states of $\mathcal{B}$, and the metric d is just the usual norm of the conjugate space $\mathcal{B}^*$. In [3] it was shown that in general, the unitary orbit does not have norm continuous local cross sections to the unitary group, but it does have local cross sections which are continuous in the quotient topology $U_\mathcal{B}/U_{\mathcal{B}^\varphi}$.

Remark 4.11. The metric $d(\Phi, \Psi) = \|\Phi - \Psi\| + \|\mathrm{supp}(\Phi) - \mathrm{supp}(\Psi)\|$ in the state space of $\mathcal{L}_\mathcal{B}(X)$ is weird. For example, in this metric, $\Sigma_{p,X}$ is open. Moreover, any state ψ of $\mathcal{L}_\mathcal{B}(X)$ such that $d(\Psi, \Sigma_{p,X}) < 1$, actually lies in $\Sigma_{p,X}$. Indeed, if Φ is a state of $\mathcal{L}_\mathcal{B}(X)$, and $d(\Phi, \varphi_x) < 1$ for some $x \in \mathcal{S}_p(X)$ and $\varphi \in \Sigma_p(\mathcal{B})$, then $\|supp(\Phi) - e_x\| < 1$, and therefore $\mathrm{supp}(\Phi)$ and e_x are unitarily equivalent. That is $\mathrm{supp}(\Phi) = e_y$, with $y = U(x)$ for some unitary U in $\mathcal{L}_\mathcal{B}(X)$. Then, by 2.2, there exists $\psi \in \Sigma_p(\mathcal{B})$ such that $\Phi = \psi_y \in \Sigma_{p,X}$.

However, if $X = \mathcal{B}$ and $\mathcal{B}$ is finite dimensional, then the topology of the d-metric coincides in $\Sigma_{p,\mathcal{B}}$ with the usual norm topology. Indeed, it suffices to see that the map $\varphi_x \mapsto e_x$ is continuous in the norm topology. Since we are in the finite dimensional case, it suffices to argue with (positive) density matrices, with trace 1. Note that the states of the form φ_x have equivalent supports, i.e. their density matrices have kernels with the same dimension. Suppose that a_n is a sequence of positive matrices with trace 1 and $nul(a_n) = k$, converging in norm to the matrix a, also with (a priori) $nul(a) = k$. Then the projections $P_{\ker a_n}$ onto the kernels converge in norm to $P_{\ker a}$. Indeed, we claim that one can find an open interval around zero and an integer n_0 such that for $n \geq n_0$ no eigenvalue of a_n (other than zero) lies inside this interval. And by a routine spectral theoretic argument, one has that $P_{\ker a_n} \to P_{\ker a}$. If one could find no such interval, then there would exist a sequence $\lambda_n > 0$ such that λ_n is an eigenvalue of a_n and $\lambda_n \to 0$. If q_n is the spectral projection corresponding to λ_n, then $a_n = b_n + \lambda_n q_n$. Then $b_n \to a$, where $nul(a) = k$ and $nul(b_n) < k$, which cannot happen.

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