A CHARACTERIZATION OF HERMITIAN MATRICES WITH VARIABLE DIAGONAL AND SMALLEST OPERATOR NORM

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Abstract. We describe properties of a Hermitian square matrix $M \in M_n(\mathbb{C})$ equivalent to that of having minimal quotient norm in the following sense:

$$\|M\| \le \|M + D\|$$

for all real diagonal matrices $D \in M_n(\mathbb{C})$ and $\| \ \|$ the operator norm. These matrices are related to some particular positive matrices with their range included in the eigenspaces of the eigenvalues $\pm\|M\|$ of M. We show how a constructive method can be used to obtain minimal matrices of any dimension relating this problem with majorization results in $\mathbb{R}^n$.

1. Introduction

Let $M_n(\mathbb{C})$ and $\mathcal{D}_n(\mathbb{R})$ be the algebras of complex and real diagonal $n \times n$ matrices. We are interested in describing Hermitian matrices $M \in M_n(\mathbb{C})$ that verify

$$\|M\| \le \|M + D\|, \text{ for all } D \in \mathcal{D}_n(\mathbb{R})$$

or equivalently

$$\|M\| = \operatorname{dist}(M, \mathcal{D}_n(\mathbb{R}))$$

(where $\| \ \|$ denotes the operator norm). These M will be called *minimal* matrices and appeared in the study of the minimal length curves in the flag manifold $\mathcal{P}(n) = \mathcal{U}(M_n(\mathbb{C}))/\mathcal{U}(\mathcal{D}_n(\mathbb{C}))$, where $\mathcal{U}(\mathcal{A})$ denotes the unitary matrices of the algebra $\mathcal{A}$. Namely, minimal curves in $\mathcal{P}(n)$ are given by action of (the class of) exponentials of anti-Hermitian minimal $n \times n$ matrices. To study anti-Hermitian minimal $n \times n$ matrices is (isometrically) equivalent to investigate the Hermitian minimal $n \times n$ matrices, and we find them notationaly simpler to consider.

The following theorem follows ideas in [3], where this problem was also studied in the context of von Neumann and C* algebras. The next result was proved in Theorem 3.3 of [1] as stated here. We write it down in its Hermitian form.

Theorem 1. *A Hermitian matrix $M \in M_n(\mathbb{C})$ is minimal in the quotient norm with respect to the diagonals if, and only if, there exists a positive semidefinite matrix $P \in M_n^h(\mathbb{C})$ such that,*

- $PM^2 = \lambda^2 P$, *where* $||M|| = \lambda$.
- *The diagonal elements of the product PM are all zero.*

Previous attempts to describe minimal matrices beyond this theorem were done in [1] in 3×3 matrices. In that work, all 3×3 minimal matrices were parametrized. However, Theorem 1 does not show how to construct $n \times n$ minimal matrices. Our goal in the present paper is to study some properties of $n \times n$ minimal matrices that allow the construction of them.

This minimal operators were studied recently in [7] where Theorem 2.2 of [1] was used to relate Leibnitz seminorms with quotient norms in C*-algebras.

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2. Preliminaries and notation

Let us call with $M_n^h(\mathbb{C})$ the set of $n \times n$ Hermitian complex matrices and with $\mathcal{D}_n(\mathbb{R})$ the subset of the diagonal real matrices. In these algebras we will denote with $\| \ \|$ the usual operator norm, that is $\|A\| = \max\{|\sigma| : \sigma \text{ is an eigenvalue of } A\}$ if $A \in M_n^h(\mathbb{C})$.

Given a matrix $A \in M_n^h(\mathbb{C})$ we will call with $\lambda(A) \subset \mathbb{R}^n$ the set of the eigenvalues of A in decreasing order and counting multiplicity, that is,

$$\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\,,$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and λ_i an eigenvalue of A. The spectrum of A will be denoted with

$$\sigma(A) = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$$

where the eigenvalues of A are listed just once and without any prescribed order.

We will denote with $\{e_i\}_{i=1}^n$ the usual canonical basis of $\mathbb{C}^n$ and with "tr" the usual trace of matrices.

Observe that if $M \in M_n^h(\mathbb{C})$ and $D \in \mathcal{D}_n(\mathbb{R})$ then $(M + D) \in M_n^h(\mathbb{C})$. Let us consider the quotient $M_n^h(\mathbb{C})/\mathcal{D}_n(\mathbb{R})$ and the quotient norm

$$|||\ [M]\ ||| = \min_{D \in \mathcal{D}_n(\mathbb{R})} \|M + D\| = \operatorname{dist}(M, \mathcal{D}_n(\mathbb{R}))$$

for $[M] = \{M + D : D \in \mathcal{D}_n(\mathbb{R})\} \in M_n^h(\mathbb{C})/\mathcal{D}_n(\mathbb{R})$. The minimum is obtained by compactness arguments.

Definition 1. *A matrix $M \in M_n^h(\mathbb{C})$ will be called* **minimal for $\mathcal{D}_n(\mathbb{R})$** *or just* **minimal** *if*

$$\|M\| \leq \|M + D\|, \quad \text{for all } D \in \mathcal{D}_n(\mathbb{R})$$

or equivalently, if $\|M\| = |||\ [M]\ ||| = \min_{D \in \mathcal{D}_n(\mathbb{R})} \|M + D\| = \mathit{dist}(M, \mathcal{D}_n(\mathbb{R}))$.

Remark 1. *Observe that if M is a minimal matrix then its spectrum is "centered" in the sense that if $\|M\| = \lambda$, then $-\lambda \in \sigma(M)$.*

For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we will denote with $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ or with $\operatorname{diag}\{a_1, a_2, \ldots, a_n\}$ the diagonal matrix of $\mathcal{D}_n(\mathbb{R})$ with $a_1, a_2, \ldots, a_n$ in the diagonal.

Given $v \in \mathbb{C}^n$, we will call with $v \otimes v$ the linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by $(v \otimes v)(x) = \langle x, v\rangle v$, for $x \in \mathbb{C}^n$ and $\langle\ ,\ \rangle$ the usual inner product in $\mathbb{C}^n$.

For $M \in M_n^h(\mathbb{C})$ and $v \in \mathbb{C}^n$ we will write $\overline{M}$ and $\overline{v}$ to denote the matrix and vector obtained from M and v by conjugation of its canonical coordinates.

If $M, N \in M_n(\mathbb{C})$ we will denote with $M \circ N$ the Schur or Hadamard product of those matrices defined by $(M \circ N)_{i,j} = M_{i,j} N_{i,j}$ for $1 \leq i, j \leq n$. Therefore, if $v \in \mathbb{C}^n$, with coordinates in the canonical basis given by $v = (v_1, v_2, \ldots, v_n)$,

$$v \circ \overline{v} = (|v_1|^2, |v_2|^2, \ldots, |v_n|^2) \in \mathbb{R}_+^n.$$

The usual matrix product will be denoted with MN, for $M, N \in M_n(\mathbb{C})$.

3. Minimal matrices

The following is a slight variation of Theorem 1.

Theorem 2. *A matrix $M \in M_n^h(\mathbb{C})$ is minimal in the quotient norm with respect to the diagonals if, and only if, there exists a positive semidefinite matrix $P \in M_n^h(\mathbb{C})$ such that,*

- *$PM^2 = \lambda^2 P$, where $||M|| = \lambda$.*
- *The diagonal elements of the product PM are all zero,*
- *P commutes with M.*

Proof. Since M is minimal if and only if the first two conditions of Theorem 2 hold for a positive P (see, Theorem 1), we only have to prove that a positive matrix P_0 that fulfills the three conditions of Theorem 2 can be chosen if M is minimal.

Suppose that the spectrum of M is $\sigma(M) = \{\lambda, -\lambda, \sigma_1, \dots, \sigma_r\}$, with $\|M\| = \lambda$ ($\lambda > |\sigma_i|$), for $1 \le i \le r$ and that Q_λ, $Q_{-\lambda}$, Q_{σ_1}, …, Q_{σ_r} are the corresponding spectral projections of M. Then,

$$M = \lambda Q_\lambda - \lambda Q_{-\lambda} + \sum_{i=1}^{r} \sigma_i Q_{\sigma_i}.$$

Observe that since $\lambda > |\sigma_i|$, for $1 \le i \le r$, then the spectral projection of M^2 for the eigenvalue λ^2 is $Q_\lambda + Q_{-\lambda}$.

Since we are supposing that M is minimal, there exists a positive semidefinite matrix P that verifies the two conditons of Theorem 1. Then, since $PM^2 = \lambda^2 P$, then P commutes with M^2. Then taking the same unitary to diagonalize P and M^2, and using that $PM^2 = \lambda^2 P$, it can be proved that $PQ = 0$ for every spectral projection Q of M^2, except the one corresponding to the eigenvalue λ^2, that is, $Q_\lambda + Q_{-\lambda}$. Therefore, the representation of P and M in blocks corresponding with the orthogonal decomposition given by the range of the orthogonal projections Q_λ, $Q_{-\lambda}$ and $I - Q_\lambda - Q_{-\lambda}$ (respectively) is

$$P = \begin{pmatrix} P_{1,1} & P_{1,2} & 0 \\ P_{1,2}^* & P_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \sum_{i=1}^{r} \sigma_i Q_{\sigma_i} \end{pmatrix}.$$

Then, using the second condition of Theorem 1, that is, $\langle PMe_i, e_i\rangle = 0$ for the canonical basis $\{e_i\}_{i=1,\dots,n}$, we obtain that

$$\langle PMe_i, e_i\rangle = \langle \begin{pmatrix} \lambda P_{1,1} & -\lambda P_{1,2} & 0 \\ \lambda P_{1,2}^* & -\lambda P_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_\lambda e_i \\ Q_{-\lambda} e_i \\ e_i - Q_\lambda e_i - Q_{-\lambda} e_i \end{pmatrix}, \begin{pmatrix} Q_\lambda e_i \\ Q_{-\lambda} e_i \\ e_i - Q_\lambda e_i - Q_{-\lambda} e_i \end{pmatrix} \rangle = 0$$

for all $i = 1, \dots, n$. Then, since $P_{1,1}Q_\lambda = P_{1,1}$, $P_{1,2}Q_{-\lambda} = P_{1,2}$, $P_{1,2}^*Q_\lambda = P_{1,2}^*$ and $P_{2,2}Q_{-\lambda} = P_{2,2}$, it follows that

$$\langle \lambda P_{1,1}e_i - \lambda P_{1,2}e_i, e_i\rangle + \langle \lambda P_{1,2}^* e_i - \lambda P_{2,2}e_i, e_i\rangle = \lambda\langle (P_{1,1} - P_{2,2})e_i, e_i\rangle + \lambda\langle (P_{1,2}^* - P_{1,2})e_i, e_i\rangle = 0$$

for all $i = 1, \dots, n$. The term $\langle (P_{1,1} - P_{2,2})e_i, e_i\rangle$ in the previous equation is real, since $P_{1,1} = Q_\lambda P Q_\lambda$ and $P_{2,2} = Q_{-\lambda} P Q_{-\lambda}$ are positive semidefinite matrices. The term $\langle (P_{1,2}^* - P_{1,2})e_i, e_i\rangle$ is purely imaginary since $\overline{\langle (P_{1,2}^* - P_{1,2})e_i, e_i\rangle} = -\langle (P_{1,2}^* - P_{1,2})e_i, e_i\rangle$. Then both terms must be zero, which implies that $\langle P_{1,1}e_i, e_i\rangle = \langle P_{2,2}e_i, e_i\rangle$. Therefore, the matrices $P_{1,1}$ and $P_{2,2}$ have the same diagonal in the canonical basis $\{e_i\}_{i=1,\dots,n}$. Then, if we define

$$P_0 = \begin{pmatrix} P_{1,1} & 0 & 0 \\ 0 & P_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

this matrix verifies

$$\langle P_0 M e_i, e_i\rangle = \lambda\left(\langle P_{1,1}e_i, e_i\rangle - \langle P_{2,2}e_i, e_i\rangle\right) = 0 \tag{3.1}$$

Moreover, $P_0 \ge 0$ and, using the block decompositions of M and P_0, it also verifies that

$$P_0 M^2 = \lambda^2 P_0 \text{ , and } P_0 M = M P_0. \tag{3.2}$$

Therefore, the equalities (3.1) and (3.2) imply that the positive semidefinite matrix P_0 verifies the three properties required. □

Remark 2. *Observe that the matrix P_0 of Theorem 2 was obtained as a diagonal block matrix in terms of the spectral projections Q_λ, $Q_{-\lambda}$, $I - Q_\lambda - Q_{-\lambda}$ of M from any matrix P verifying Theorem 1.*

The proof of Theorem 2 suggests another equivalent condition for being minimal:

Corollary 1. *Given a matrix $M \in M_n^h(\mathbb{C})$ the following statements are equivalent:*

- *M is minimal*
- *$\pm\|M\| \in \sigma(M)$ and there exist a pair positive semidefinite matrices $P_+, P_- \in M_n^h(\mathbb{C})$, such that, if $Q_{\|M\|}$, $Q_{-\|M\|}$ are the spectral projections of M with respect to the eigenvalues $\pm\|M\|$ respectively, they satisfy the following*
  i) $P_+ Q_{\|M\|} = Q_{\|M\|} P_+ = P_+$
  ii) $P_- Q_{-\|M\|} = Q_{-\|M\|} P_- = P_-$
  iii) $\langle P_- e_i, e_i\rangle = \langle P_+ e_i, e_i\rangle$, *for all e_i, $i = 1, \dots, n$, the canonical basis of $\mathbb{C}^n$.*

Proof. If we suppose that M is minimal it suffices to choose $P_+ = P_{1,1}$ and $P_- = P_{2,2}$ from the proof of Theorem 2.

If there exist such P_+ and P_- then a direct calculation shows that the matrix $P = P_+ + P_-$ fulfills the requirements of Theorem 2, and therefore M is minimal. □

This corollary motivates the following definition.

Definition 2. *Given a positive semidefinite matrix $P \in M_n^h(\mathbb{C})$, another positive semidefinite $Q \in M_n^h(\mathbb{C})$ is called a* **companion** *matrix of P if, $PQ = 0$ (being 0 the null matrix) and they both have the same diagonal in the canonical basis. We will say that P has a companion Q, or that P and Q are companions.*

Remark 3. *i) Note that if P is a companion of Q, then Q is a companion of P.*

ii) If P and Q are companions then they must have the same trace since they have the same diagonal.

iii) If P is a companion of Q and $P \neq 0$, then $Q \neq 0$. This holds because if $Q = 0$ then the diagonal of P must be zero in the canonical basis. This yields to $P = 0$ since P is positive semidefinite, a contradiction. Therefore, if P and Q are companions and one of them is 0, then the other must be 0.

iv) Observe that not every positive semidefinite matrix P has a companion. For example, if P is invertible, then it has not got any companion matrix. Therefore, if a matrix P has a companion, then P must have non trivial kernel.

v) Note that a matrix P could have many companions. Take por example any 3×3 complex Hadamard matrix H (that is a matrix such that $|H_{i,j}| = 1$ with orthogonal rows and columns), and consider the unitary matrix $U = \frac{1}{\sqrt{3}}H$. Then, if $diag(a,b,c)$ denotes the 3×3 diagonal matrix with a, b and c in its diagonal, and we define $P = U\,diag(4,0,0)\,U^*$ and $Q_t = U\,diag(0,4-t,t)\,U^*$ for $t \in \mathbb{R}$ and $0 \le t \le 4$, an easy check proves that $\{Q_t\}_{0\le t\le 4}$ are all different companion matrices of P.

In the following corollary, if $Q \in M_n(\mathbb{C})$, then $\operatorname{ran}(Q)$ will denote the range of the corresponding linear transformation.

Corollary 2. *Given S_1, S_2 subspaces of $\mathbb{C}^n$ with $S_1 \perp S_2$, then the following statements are equivalent:*

i) *There exist positive semidefinite matrices $P_1, P_2 \in M_n^h(\mathbb{C})$, with ran$(P_1) \subset S_1$ and ran$(P_2) \subset S_2$, such that P_1 and P_2 are companions.*
ii) *$M = \lambda\, P_{S_1} - \lambda\, P_{S_2} + R$ is a minimal matrix, for every $\lambda > 0$ and $R \in M_n^h(\mathbb{C})$ such that $P_{S_1}R = P_{S_2}R = 0$ and $\|R\| < \lambda$ (with P_{S_1} and P_{S_2} the respective orthogonal projections onto the subspaces S_1 and S_2).*

Proof. Let us suppose first that P_1 and P_2 are companion matrices with the hypothesis of i). Consider then $\lambda > 0$ and a matrix $M = \lambda\, P_{\operatorname{ran}(P_1)} - \lambda\, P_{\operatorname{ran}(P_2)} + R$, with R such that its range is orthogonal to that of P_1 and P_2 and $\|R\| < \lambda$. Then taking $P = P_1 + P_2$ it is easy to verify that P and M satisfy the conditions of Theorem 1 that imply that M is minimal with $\|M\| = \lambda$.

Let us suppose now that $M = \lambda\, P_{S_2} - \lambda\, P_{S_2} + R$ as in item ii) is a minimal matrix. Then using that $S_1 \perp S_2$, that $\operatorname{ran}(R)$ is orthogonal to $S_1 \oplus S_2$ and that $\|R\| < \lambda$, it is apparent that the spectral projections Q_λ, $Q_{-\lambda}$ of M with respect to the eigenvalues λ and $-\lambda$ verify that $Q_\lambda = P_{S_1}$ and $Q_{-\lambda} = P_{S_2}$. Then there exists a positive semidefinite $P \in M_n^h(\mathbb{C})$ that verifies the three statements of Theorem 2. Therefore

P commutes with M. As in the proof of Theorem 2 it can be proved that the representation of P as a block matrix with respect to the orthogonal subspaces S_1, S_2 and $(S_1 \oplus S_2)^\perp$ is

$$P = \begin{pmatrix} P_{1,1} & 0 & 0 \\ 0 & P_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We shall prove that $P_1 = P_{1,1} = P_{S_1} P P_{S_1}$ and $P_2 = P_{2,2} = P_{S_2} P P_{S_2}$ fulfill the conditions of i). Since P is positive semidefinite it is apparent that P_1 and P_2 are also positive semidefinite. By definition, $\operatorname{ran}(P_1) \subset S_1$ and $\operatorname{ran}(P_2) \subset S_2$ and $P_1 P_2 = 0$.

Moreover, since PM has zero diagonal in the canonical basis, then

$$PM = \begin{pmatrix} \lambda P_1 & 0 & 0 \\ 0 & -\lambda P_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has zero diagonal in the canonical basis of $\mathbb{C}^n$. That means that $\lambda\langle (P_1 - P_2)e_i, e_i\rangle = 0$ for the canonical basis $\{e_i\}_{i=1,\dots,n}$ of $\mathbb{C}^n$, and then the diagonals of P_1 and P_2 coincide in that basis. Therefore, P_1 is a companion of P_2. □

4. Characterization of companion matrices

Corollary 2 gives a direct relation between minimal matrices and pairs of companion matrices. Moreover, if one has a pair of companion matrices then a minimal matrix can be constructed as in ii) of that corollary. In this section we will describe some of the properties of the companion matrices.

Recall that, as it was mentioned in the preliminaries, for a given vector $v \in \mathbb{C}^n$,

$$v \circ \overline{v} = (|v_1|^2, |v_2|^2, \dots, |v_n|^2) = \sum_{j=1}^{n} |v_j|^2 e_i \in \mathbb{R}^n_+,$$

if v has canonical coordinates $(v_1, v_2, \dots, v_n)$. For given vectors $\{w_k\}_{k=1}^m \subset \mathbb{C}^n$ we will denote with $\mathrm{K}\left(\{w_k\}_{k=1}^m\right)$ and $\mathrm{co}\left(\{w_k\}_{k=1}^m\right)$ the cone and the convex hull generated by them (respectively).

Theorem 3. *Let $P \in M_n^h(\mathbb{C})$ be a positive semidefinite matrix, its eigenvalues counted with multiplicity given by $\lambda(P) = (a_1, a_2, \dots, a_r, 0, \dots, 0)$, with $a_i > 0$, $1 \le r < n$. Then the following properties of P are equivalent*

i) *P has a companion Q.*

ii) *There exist a set of orthonormal eigenvectors $\{v_1, v_2, \dots, v_r\}$ corresponding to the (strictly) positive eigenvalues $a_1, a_2, \dots a_r$ of P and another set of orthonormal eigenvectors $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ of the kernel of P, and $x_j \ge 0$ such that*

$$\sum_{i=1}^{r} a_i (v_i \circ \overline{v_i}) = \sum_{j=r+1}^{n} x_j (v_j \circ \overline{v_j}). \tag{4.1}$$

iii) *There exist a set of orthonormal eigenvectors $\{v_1, v_2, \dots, v_r\}$ corresponding to the (strictly) positive eigenvalues $a_1, a_2, \dots a_r$ of P and another set of orthonormal eigenvectors $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ of the kernel of P such that*

$$\sum_{i=1}^{r} a_i (v_i \circ \overline{v_i}) \in K\left(\{v_j \circ \overline{v_j}\}_{j=r+1}^{n}\right).$$

iv) *There exists a set of orthonormal eigenvectors $\{v_i\}_{i=1}^r$ of P corresponding to the (strictly) positive eigenvalues $a_1, a_2, \dots a_r$ of P and orthogonal eigenvectors $\{v_j\}_{j=r+1}^{r+s} \subset Ker(P)$, that verify*

$$\sum_{i=1}^{r} \frac{a_i}{tr(P)} v_i \circ \overline{v_i} \in co\left(\{v_j \circ \overline{v_j}\}_{j=r+1}^{r+s}\right).$$

Proof. Let us suppose first that P has a companion Q, and the spectrum of P, counting multiplicity of eigenvalues and in descending order, is $\lambda(P) = (a_1, a_2, \dots, a_r, 0, \dots, 0)$, with $a_r > 0$. Then, since $PQ = 0$, they commute, and therefore we can choose a unitary matrix V that diagonalizes both P and Q. We can also choose V in the following way:

$$V = \begin{pmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,n} \\ v_{2,1} & v_{2,2} & \dots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \dots & v_{n,n} \end{pmatrix} \tag{4.2}$$

where the columns are the coordinates in the canonical basis of $\mathbb{C}^n$ of an orthonormal basis $\{v_i\}_{1\le i\le n}$ of eigenvectors of P and $v_i = (v_{1,i}, v_{2,i}, \dots, v_{n,i})$ is the corresponding eigenvector of a_i (for $1 \le i \le r$). Then, this V verifies that $P = VD_PV^*$ and $Q = VD_QV^*$, where D_P is a diagonal matrix with $\lambda(P)$ in its diagonal and D_Q is a diagonal with the eigenvalues of Q in its diagonal. Since Q must be positive and $PQ = 0$, then the diagonal of D_Q has to be of the form $\{0, 0, \dots, 0, x_{r+1}, x_{r+2}, \dots, x_n\}$ with $x_i \ge 0$, for $r + 1 \le i \le n$. Moreover, since P and Q have identical diagonals in the canonical basis, then considering the decompositions

$$P = VD_PV^* =$$

$$= \begin{pmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,n} \\ v_{2,1} & v_{2,2} & \dots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \dots & v_{n,n} \end{pmatrix} \cdot \begin{pmatrix} a_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & a_2 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & a_r & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \cdot \begin{pmatrix} \overline{v_{1,1}} & \overline{v_{2,1}} & \dots & \overline{v_{n,1}} \\ \overline{v_{1,2}} & \overline{v_{2,2}} & \dots & \overline{v_{n,2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{v_{1,n}} & \overline{v_{2,n}} & \dots & \overline{v_{n,n}} \end{pmatrix}$$

and

$$Q = VD_QV^* =$$

$$= \begin{pmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,n} \\ v_{2,1} & v_{2,2} & \dots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \dots & v_{n,n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & x_{r+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & x_n \end{pmatrix} \cdot \begin{pmatrix} \overline{v_{1,1}} & \overline{v_{2,1}} & \dots & \overline{v_{n,1}} \\ \overline{v_{1,2}} & \overline{v_{2,2}} & \dots & \overline{v_{n,2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{v_{1,n}} & \overline{v_{2,n}} & \dots & \overline{v_{n,n}} \end{pmatrix}$$

we obtain the n following equations

$$\begin{cases} \sum_{i=1}^r a_i |v_{1,i}|^2 & = & \sum_{j=r+1}^n x_j |v_{1,j}|^2 \\ \sum_{i=1}^r a_i |v_{2,i}|^2 & = & \sum_{j=r+1}^n x_j |v_{2,j}|^2 \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^r a_i |v_{n,i}|^2 & = & \sum_{j=r+1}^n x_j |v_{n,j}|^2 \end{cases}.$$

Then,

$$\left(\sum_{i=1}^{r} a_i|v_{1,i}|^2, \sum_{i=1}^{r} a_i|v_{2,i}|^2, \dots, \sum_{i=1}^{r} a_i|v_{n,i}|^2\right) = \left(\sum_{j=r+1}^{n} x_j|v_{1,j}|^2, \sum_{j=r+1}^{n} x_j|v_{2,j}|^2, \dots, \sum_{j=r+1}^{n} x_j|v_{n,j}|^2\right)$$

and

$$\sum_{i=1}^{r} a_j \left(|v_{1,i}|^2, |v_{2,i}|^2, \dots, |v_{n,i}|^2\right) = \sum_{j=r+1}^{n} x_j \left(|v_{1,j}|^2, |v_{2,j}|^2, \dots, |v_{n,j}|^2\right) \tag{4.3}$$

which proves ii).

Now suppose that ii) holds. If we define $Q = \sum_{j=r+1}^{n} x_j \ (v_j \otimes v_j)$, with x_j and v_j as in ii), then it verifies that $PQ = 0$. Moreover, since the equality (4.1) is equivalent to the equality of the diagonals of P and Q, then Q is a companion of P.

Assertion iii) is equivalent to ii) since $\sum_{j=r+1}^{n} x_j(v_j \circ \overline{v_j})$ is a generic element of the cone generated by $\{v_j \circ \overline{v_j}\}_{j=r+1}^{n}$.

Statement ii) implies iv) because the equality (4.1) is equivalent to the fact that P has the same diagonal than $Q = \sum_{j=r+1}^{r+s} x_j \ (v_j \otimes v_j)$. Then P and Q have the same trace equal to $\sum_{i=1}^{r} a_i = \sum_{j=r+1}^{r+s} x_j$. Therefore

$$\sum_{j=1}^{r} \frac{a_j}{\sum_{i=1}^{r} a_i} v_j \circ \overline{v_j} = \sum_{j=r+1}^{n} \frac{x_j}{\sum_{j=r+1}^{r+s} x_j} v_j \circ \overline{v_j} \ \in \ \mathrm{co}\left(\{v_j \circ \overline{v_j}\}_{j=r+1}^{r+s}\right).$$

If iv) holds then obviously iii) an ii) hold.

□

Considering the results obtained in Corolllary 2 and Theorem 3 we can conclude that a matrix $M = \lambda P_{S_1} - \lambda P_{S_2} + R \in M_n^h(\mathbb{C})$ (with $S_1 \perp S_2$ and $R \in M_n^h(\mathbb{C})$ with $\|R\| < \lambda$) is minimal, if and only if, there exist orthonormal vectors $\{v_i\}_{i=1}^{r} \subset S_1$ and $\{v_j\}_{j=r+1}^{r+s} \subset S_2$ such that

$$\mathrm{co}\big(\{v_i \circ \overline{v_i}\}_{i=1}^{r}\big) \cap \mathrm{co}\left(\{v_j \circ \overline{v_j}\}_{j=r+1}^{r+s}\right) \neq \emptyset.$$

Note also that any minimal matrix is necessarily of this form.

Moreover, given a matrix $M \in M_n^h(\mathbb{C})$, then M is minimal, if and only if, there exists a unitary matrix U such that $U^*MU = \mathrm{diag}\,(\lambda(M))$ and the rows of the unistochastic matrix $U^* \circ \overline{U^*}$ have the required properties with respect to the eigenspaces of $\lambda = \|M\|$ and $-\lambda$ of M. Namely, that

$$\mathrm{co}\left(\{v_i \circ \overline{v_i}\}_{i=1}^{r}\right) \cap \mathrm{co}\left(\{v_j \circ \overline{v_j}\}_{j=r+1}^{r+s}\right) \neq \emptyset,$$

where $\{v_i\}_{i=1}^{r}$ are the corresponding orthogonal eigenvectors of λ (and rows of U) and $\{v_j\}_{j=r+1}^{r+s}$ are the corresponding orthogonal eigenvectors of $-\lambda$ (and rows of U).

Observe that following the notation of Theorem 3 ii), since $\sum_{i=1}^{r} a_i = \sum_{j=r+1}^{n} x_j$, then

$$(0, 0, \dots, 0) \prec (a_1, \dots, a_r, -x_{r+1}, \dots, -x_n) = \vec{ax}$$

(where $\prec$ is the usual notation for majorization of vectors in $\mathbb{R}^n$, see [5]). Then the equations in (4.3) prove that the matrix

$$V^* \circ \overline{V^*} = \begin{pmatrix} |v_{1,1}|^2 & |v_{2,1}|^2 & \dots & |v_{n,1}|^2 \\ |v_{1,2}|^2 & |v_{2,2}|^2 & \dots & |v_{n,2}|^2 \\ \vdots & \vdots & \ddots & \vdots \\ |v_{1,n}|^2 & |v_{2,n}|^2 & \dots & |v_{n,n}|^2 \end{pmatrix}$$

obtained from the matrix (4.2) is a doubly stochastic (in fact, unistochastic) matrix that verifies $(0, \dots, 0) = \vec{ax}.\left(V^* \circ \overline{V^*}\right)$. This suggests a relation with results in majorization of vectors in $\mathbb{R}^n$.

Take any n-tuple $a\vec{0}x = (a_1, \ldots, a_r, 0, \ldots, 0, -x_1, \ldots, -x_s) \in \mathbb{R}^n$, with $a_i \geq 0$ and $x_j \geq 0$, such that $\sum_{i=1}^r a_i = \sum_{j=1}^s x_j$. Then it is apparent that $(0, \ldots, 0) \prec a\vec{0}x$. Therefore a concrete unitary or orthogonal matrix U can be found (see [4, 6]) such that $(0, \ldots, 0) = a\vec{0}x.(U \circ \overline{U})$. Then, if we call with v_k the k-th column of U^* (for $k = 1, \ldots, n$), any matrix of the form

$$M = \lambda \sum_{i=1}^{r} v_i \otimes v_i + \sum_{h=r+1}^{n-s} \lambda_h (v_h \otimes v_h) - \lambda \sum_{j=n-s+1}^{n} v_j \otimes v_j \tag{4.4}$$

is minimal provided that $\lambda > 0$, $\lambda_h \in \mathbb{R}$ and $|\lambda_h| < \lambda$. These results, together with Corollary 2 and Theorem 3 allow to construct minimal matrices of any size.

The method to obtain minimal matrices M mentioned in (4.4) relies on which is the unitary U retrieved from the unistochastic matrix. The work of [2] shows different algorithms to find such a unitary or even orthogonal matrix U that verifies $\overrightarrow{0} = a\vec{0}x.U \circ \overline{U}$. Nevertheless, the set of all posible unitaries U that give the same unistochastic matrix is not known in general. The works of [8] and [9] study the problem of describing the different matrices U such that the mapping $U \mapsto U \circ \overline{U}$ gives the same unistochastic matrix.

Remark 4. *In* [1] *a different characterization of minimal* 3×3 *matrices were given. It was shown that given a* 3×3 *matrix* M*, with* $\lambda(M) = (\lambda, \mu, -\lambda)$*,* $|\mu| \leq \lambda = \|M\|$*, then,* M *was minimal, if and ony if, there exists an orthonormal eigenvector* v_λ *of the eigenvalue* λ *and a orthonormal eigenvector* $v_{-\lambda}$ *of the eigenvalue* $-\lambda$ *such that* $v_\lambda \circ \overline{v_\lambda} = v_{-\lambda} \circ \overline{v_{-\lambda}}$*. The statement remains valid if any of the eigenvalues has multiplicity two* $(\mu = \pm\lambda)$*. The following is an example of a* 4×4 *minimal Hermitian matrix where this condition does not hold. Let*

$$M = \begin{pmatrix} \frac{9}{14} & -\frac{15}{14} - \frac{i}{7} & -\frac{1}{7} + \frac{5i}{7} & \frac{2}{7} + \frac{6i}{7} \\ -\frac{15}{14} + \frac{i}{7} & \frac{13}{14} & -\frac{1}{7} + i & \frac{6i}{7} \\ -\frac{1}{7} - \frac{5i}{7} & -\frac{1}{7} - i & \frac{5}{7} & -1 - \frac{2i}{7} \\ \frac{2}{7} - \frac{6i}{7} & -\frac{6i}{7} & -1 + \frac{2i}{7} & \frac{5}{7} \end{pmatrix}.$$

Then $\lambda(M) = (2, 2, 1, -2)$*, and the eigenspace of the eigenvalue* 2 *is generated by the orthonormal eigenvectors* $v_1 = \frac{1}{5\sqrt{2}}(-1-2i, 5, -3-i, 1-3i)$ *and* $v_2 = \frac{1}{10\sqrt{14}}(17-11i, -15+5i, -9+17i, 3-19i)$*. The vector* $w = \frac{1}{2\sqrt{2}}(1-i, 1-i, 1+i, 1+i)$ *is a orthonormal eigenvector of eigenvalue* -2*. A direct calculation shows that for* $\alpha = \frac{2}{9}$*, then* $\alpha(v_1 \circ \overline{v_2}) + (1-\alpha)(v_2 \circ \overline{v_2}) = w \circ \overline{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$*, which is enough to prove that* M *is minimal (using Theorem 3 and Corollary 2). Nevertheless, there is not a single eigenvector* v *in the eigenspace of* λ *such that* $v \circ \overline{v} = w \circ \overline{w}$*. This follows after writing* $v = \beta v_1 + \gamma v_2$ *with* $\beta, \gamma \in \mathbb{C}$*, and* $|\beta|^2 + |\gamma|^2 = 1$*, and proving that* $v \circ \overline{v} = w \circ \overline{w}$ *could never happen (note that we can suppose that* $\gamma = \sqrt{1 - |\beta|^2}$*).*

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